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The double obstacle problem and the two membranes problem

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The double obstacle problem and the two membranes problem

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Esta tesis está dedicada a mis padres,

Jorge Alberto Duque y Maria Cristina Álvarez

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The double obstacle problem and the two membranes problem

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In the first part of this dissertation, we study the existence, regularity and the free boundary of the double obstacle problem in different formulations that involve linear, elliptic, parabolic and fully nonlinear equations.

The second part focuses on the two membranes problem for fully nonlinear elliptic operators, here we prove the existence of solutions and then we prove the optimal regularity when the operators involved are the Pucci extremal operators. Finally, we give an example that shows that no regularity for the free boundary is to be expected to hold in general.

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Chapter 1

Introduction

1.1 The obstacle problem

Even though most of the developments outlined in this section were discovered many decades ago, the simplicity of this setting and the deepness of the results are still the motivation for many free boundary problems nowadays.

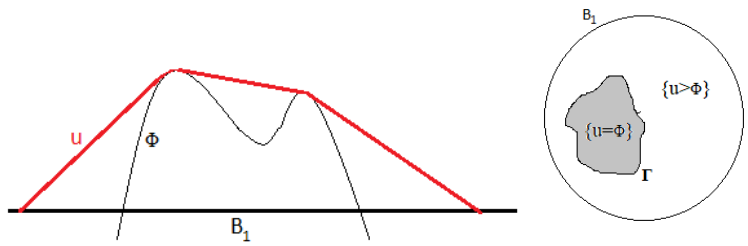
The classic obstacle problem consists of studying the equilibrium position of an elastic membrane denoted by u that lies above a fixed obstacle ϕ. Figure 1.1 (a) shows the solution of the obstacle problem in one dimension. Formally one tries to find a function

$$u \in \mathbb{K} := \{w : B_1 \to \mathbb{R} \ \mid \ w \geq \phi \text{ on } B_1 \text{ and } \ u = g \text{ on } \partial B_1\}$$

that minimizes the Dirichlet energy defined as

$$I[w] := \int_{B_1} |\nabla w|^2 \tag{1.1}$$

Where B_1 denotes the ball of dimension n. If we knew that the membrane is not touching the obstacle, we would know that u is harmonic in B_1, that is, we would have $\Delta u = 0$ in B_1 and it is known (see for instance [10]) that all the derivatives of harmonic functions are continuous, that is, $u \in C^\infty(B_1)$.



(a) The obstacle problem in one dimension (b) The free boundary

Figure 1.1: The obstacle problem

In a scenario where u actually touches the obstacle, the situation becomes more complicated. In this case, instead of being harmonic, u satisfies

$$\begin{cases} u \geq \phi & \text{in} \quad B_1 \\ \Delta u \leq 0 & \text{in} \quad B_1 \\ \Delta u = 0 & \text{in} \quad \{u > \phi\} \cap B_1 \\ u = g & \text{on} \quad \partial B_1 \end{cases} \tag{1.2}$$

and, as u solves an equation ($\Delta u = 0$) in a domain that is not known a priori ($\{u > \phi\} \cap B_1$) we say that this is a *free boundary problem*; Figure 1.1 (b) shows the set in which the membrane u touches the obstacle in a two dimensional obstacle problem. We define then the non-contact set $A := \{u > \phi\}$, the contact set $\Lambda := \{u = \phi\}$ and the free boundary $\Gamma := B_1 \cap \partial A$. In this case, we are mainly interested on two questions: How regular is the solution? and, how regular is the free boundary?. these two questions can also be asked for the problems studied in this thesis and were our main motivation. Some of the main results known for the obstacle problem are:

1. Existence of solutions: using the direct method of the calculus of variations (see [10]) it is possible to show that u exists, it is unique and it

has weak derivatives of first order, that is $u \in H^1(B_1)$

2. Optimal regularity of the solution: Frehse and Mosco showed in [11] that u is as regular as ϕ up to $C^{1,1}$; in particular if ϕ is smooth then u has continuous Lipschitz derivatives. This is known to be the best possible regularity one can expect since even in the one dimensional case, it is possible to construct solutions with discontinuous second derivatives.

3. Regularity of the free boundary: Caffarelli showed in [2] that the free boundary is locally a $C^{1,\alpha}$ manifold of dimension $(n-1)$ around regular points [1].

4. Structure of the singular set: Caffarelli showed in [5] that if $x_0 \in \Gamma$ is a singular point (i.e. a point that is not regular) then the set of singular points of Γ lies in a k-dimensional C^1 manifold around x_0 where the dimension k depends on our choice of x_0.

5. Higher order regularity of the free boundary: Kinderlehrer and Nirenberg showed in [17] that the free boundary is locally a smooth manifold (i.e. $\Gamma \in C^\infty$) around regular points.

The problems and results studied in the following chapters of this dissertation resemble in many ways the obstacle problem. Moreover, our initial

[1]More precisely, Caffarelli showed that there exist a universal modulus of continuity $\rho(r)$ so that if $x_0 \in \Gamma$ and $B_r(x_0) \cap \Lambda$ cannot be enclosed by any strip of width $r\rho(r)$ then the free boundary is a $C^{1,\alpha}$ manifold of dimension (n-1) around x_0; in this case x_0 is called a regular point

goals in every single scenario (even though this was not always achieved) was to study the existence, regularity of the solution and regularity of the free boundary and obtain similar results to those described in the previous settings.

1.2 The classic double obstacle problem

The classic double obstacle problem studies the equilibrium position of an elastic membrane that lies between two fixed obstacles. If it is the case that the membrane (u) does not touch the upper obstacle then u will be a solution of the classic obstacle problem, that is u will solve Equation (1.2). In general, if u does touch both membranes it will satisfy

$$\begin{cases} \phi_1 \leq u \leq \phi_2 & \text{in} \quad B_1 \\ \Delta u \leq 0 & \text{in} \quad B_1 \cap \{u < \phi_2\} \\ \Delta u \geq 0 & \text{in} \quad B_1 \cap \{u > \phi_1\} \\ u = g & \text{on} \quad \partial B_1 \end{cases} \tag{1.3}$$

There are several situations in which Equations similar to Equation 1.3 appear but operators different to the Laplacian are involved. Consider for instance a membrane u that lies between two obstacles $\phi_1 < \phi_2$ and instead of minimizing the Dirichlet Energy (Equation 1.1), u minimizes

$$J[w] := \int_{B_1} F(\nabla w) \tag{1.4}$$

for some convex function F. In this scenario u will satisfy Equation 1.3 but instead of the Laplacian we will have operator in divergence form involved (i.e. a term of the form $\nabla \cdot \big(\nabla F(\nabla u)\big)$ instead of Δu)

In Chapter 2 we study the double obstacle problem in several different situations motivated by optimal stopping problems; we study a particular scenario known in the probability literature as a 'Dynkin game' (see [9]). In these scenarios, u represents the optimal way of playing a game and will satisfy Equations similar to Equation 1.3 except that the involved operator will not be the Laplacian but will be a linear elliptic operator in nondivergence form, a fully nonlinear elliptic operator or parabolic operator instead.

1.3 The classic two membranes problem

The two membranes problem consists of studying the position of equilibrium of two elastic membranes (possibly made of different materials) that are on top of each other and touch in a region, that is, we want to find we want to find a pair of functions (u, v) defined in the ball B_1 with $u \leq v$ that minimize the energy

$$J[u,v] := \int_{B_1} F(\nabla u) + \int_{B_1} F(\nabla v) + \int_{B_1} fu + \int_{B_1} gv \tag{1.5}$$

In this scenario v solves a (lower) obstacle problem similar to Equation (1.2) with obstacle u and u solves an (upper) obstacle problem with obstacle v for a certain elliptic operator.

Similar to what we explained in the previous section, there are several situations in which the involved operator is not going to be the Laplacian anymore; in Chapter 3, motivated by an optimal stopping scenario, we study the two membranes problem for two different second order differential operators.

Chapter 2

The Double Obstacle Problem

The literature on the elliptic single obstacle problems is vast. The optimal regularity of the solution and a detailed study of the free boundary can be found in [5] in the case of the Laplace operator. In [7], Kinderlehrer and Caffarelli studied the solution of this problem for elliptic operators with variable coefficients.

In [24] and [22], Petrosyan and Shahgholian studied the regularity of the solution and the free boundary in nondivergence form of the parabolic single obstacle in different scenarios, including operators with constant coefficients and fully nonlinear elliptic ones. They also presented the relation between these problems and the study of American options and choose their obstacles accordingly.

The initial motivation of this Chapter is to generalize the results of [5], [24] and [22] to situations involving two obstacles, elliptic and parabolic operators. The double obstacle problem studies the equilibrium position of an elastic membrane that lies between two fixed obstacles. In this case, if the solution does not touch one of the obstacles it will be a solution to a single obstacle problem.

The regularity for this problem was initially studied by Dal Maso, Mosco and Vivaldi; in [13] they studied the double obstacle problem with constant coefficients in the context of variational inequalities motivated by the work of Brezis, Lewi and Stampachia (see [18]). They showed that the solution is continuous even when the obstacles are irregular as long as they satisfy a Wiener-type condition.

Soon after this, Kilpelainen and Ziemer studied the situation for nonlinear elliptic variational inequalities. In this case, the differential operator associated with the solution is quasi-linear and includes operators with bounded coefficients in divergence form. They showed that when the obstacles are Hölder continuous, the solution of the problem is also Hölder continuous. Moreover, they proved that even for a family of very irregular obstacles (that can, for instance, have discontinuities) the solution remains continuous.

Motivated by an optimal stopping situation coming from financial options, in this chapter we study the double obstacle problem in nondivergence form; this type of scenario was originally proposed by Dynkin in [9] from a probabilistic and discrete point of view.

2.1 The linear elliptic case

Suppose that we have a particle (or, for instance, the price of several assets) that moves inside B_1 at each instant of time in the following way: if the particle is in the position x at time $t = t_0$ then, the location of the particle at $t = t_0 + \Delta_t$ will be given by a uniform distribution in the region

$E_{A(x)}(x)$. Notice that given an elliptic matrix A, $E_A(y)$ denotes an ellipsoid with ellipticity A centered at y.

Suppose now that there is a *player* that is playing a game against the *house* with the following rules:

1. There is a 'payoff' function $\phi_1(x)$ (with $\phi_1 : B_1 \to R$) and the player can stop the game at any instant of time and leave with an amount of money given by $\phi_1(x)$. That is, if in a certain instant of time the particle is located at the position x then the player can leave the game with a profit of $\phi_1(x)$.

2. The house can stop the game at any moment. If this is the case, the player will leave with an amount of money given by $\phi_2(x)$ (with $\phi_2 : B_1 \to R$ and $\phi_2 > \phi_1$) where x denotes the position of the particle at that instant of time.

3. Whenever the particle touches the border of the ball the game stops and the player leaves with a prescribed amount of money given by $g(x)$ where $g : \partial B_1 \to R$.

Let $u(x)$ be the expected value that the player of this game is going to make if the particle was initially located at x, then u satisfies

$$\begin{cases} \phi_1 \leq u \leq \phi_2 & \text{in} \quad B_1 \\ u = g & \text{on} \quad \partial B_1 \\ tr(AD^2u) \leq 0 & \text{in} \quad \{u < \phi_2\} \cap B_1 \\ tr(AD^2u) \geq 0 & \text{in} \quad \{u > \phi_1\} \cap B_1 \end{cases} \tag{2.1}$$

where $A = A(x)$ is a uniformly elliptic matrix with $\lambda \leq A \leq \Lambda$. In the following subsections we study the existence and regularity of a function u satisfying 2.1. In this section we denote $F(M,x) := tr(A(x)M)$.

2.1.1 Existence

In order to show existence of a function u satisfying Equation 2.1 we follow a penalization technique widely used in the books of Kinderlehrer and Stampachia ([18]) and Friedman ([12]). First we study the family of penalized equations

$$\begin{cases} F(D^2u^\epsilon) = \beta_\epsilon(u^\epsilon - \phi_1) - \beta_\epsilon(\phi_2 - u^\epsilon) & \text{on } B_1 \\ u^\epsilon = g & \text{on } \partial B_1 \end{cases} \tag{2.2}$$

Where, for each $\epsilon > 0$, $\beta_\epsilon : \mathbb{R} \to \mathbb{R}$ is a smooth function with the following properties:

- $\lim_{s\to-\infty} \beta_\epsilon(s) = -\infty$
- $\beta'_\epsilon := \partial\beta_\epsilon > 0$
- $\beta_\epsilon(0) = -C$
- $-C \leq \beta_\epsilon(s) \leq C$ for every $s > 0$
- $\lim_{\epsilon\to 0} \beta_\epsilon(s) = 0$ for every $s > 0$
- $\lim_{\epsilon\to 0} \beta_\epsilon(s) = -\infty$ for every $s < 0$

Our goal is to show that u^ϵ converges to the solution of our elliptic problem u when $\epsilon \to 0$, first we need some lemmas.

Lemma 2.1.1. *Let $\phi_1, \phi_2 \in C^2$. There exist a solution $u^\epsilon \in W^{2,p}(B_1)$ to the penalized problem of Equation 2.2, moreover $||u^\epsilon||_{W^{w,p}} \leq C$ where C is a constant that does not depend on ϵ*

Proof. Fix $\epsilon > 0$, for each $N \in \mathbb{N}$ let β_ϵ^N a truncation of β_ϵ between N and $-N$ (i.e. $\beta_\epsilon^N := \max\{\min\{\beta_\epsilon, N\}, -N\}$), regularize β_ϵ^N if necessary making sure that $\partial\beta_\epsilon^N > 0$ still holds.

We define the mapping $T : W^{2,p}(B_1) \to W^{2,p}(B_1)$ with $T(w) = v$ where v is the solution to the problem

$$\begin{cases} F(D^2 v) = \beta_\epsilon^N(w - \phi_1) - \beta_\epsilon^N(\phi_2 - w) & \text{on} \quad B_1 \\ \qquad v = g & \text{on} \quad \partial B_1 \end{cases} \tag{2.3}$$

To see that this map is well defined notice that the right hand side of Equation 2.4 is in $L^p(B_1)$ since β_ϵ^N is bounded and hence from the Calderon Zygmund estimates we have that $u \in W^{2,p}(B_1)$. Moreover from these estimates we have that $||T(u)||_{W^{2,p}} = ||v||_{W^{2,p}} \leq CN$ for some universal constant $C > 0$ and hence we can find a radius $R > 0$ so that $T(B_R) \subset B_R$. From the previous observation and Schauder's fixed point theorem we conclude that there exists $u^N \in W^{2,p}$ such that $T(u^N) = u^N$, that is:

$$\begin{cases} F(D^2 u^N) = \beta_\epsilon^N(u^N - \phi_1) - \beta_\epsilon^N(\phi_2 - u^N) & \text{on} \quad B_1 \\ \qquad u^N = g & \text{on} \quad \partial B_1 \end{cases} \tag{2.4}$$

We now want to show that if N is sufficiently big then $u^N = u^\epsilon$ is a solution of Equation 2.2. We first show that the L^∞ norm of the right hand side of 2.4 does not depend on N and ϵ. To do this let $x \in B_1$, define $\eta := \beta_\epsilon^N(u^N - \phi_1) - \beta_\epsilon^N(\phi_2 - u^N)$ and notice

Observation 1: $\eta_1 := \beta_\epsilon^N(u^N - \phi_1)$ is uniformly bounded by below. To see this, let y a minimum of η_1. If $y \in \partial B_1$ from the compatibility condition (i.e. $\phi_1 < g < \phi_2$ in ∂B_1) we would be done. If $y \in B_1$ we have

$$\beta_\epsilon^N(u^N(y)-\phi_1(y)) = F(D^2(u^N(y)-\phi_1(y)))+\beta_\epsilon^N(\phi_2(y)-u^N(y))+F(D^2\phi_1(y))$$

And, as β_ϵ^N is monotone we know that $u^N - \phi_1$ also achieves a minimum at y and hence

$$\beta_\epsilon^N(u^N(y) - \phi_1(y)) \geq 0 + \beta_\epsilon^N(\phi_2(y) - u^N(y)) + F(D^2\phi_1(y)) \tag{2.5}$$

Now notice that if $u^N(y) - \phi_1(y) \geq 0$ we would have immediately from the definition of β_ϵ^N that $\eta_1(y) \geq 0$, so let's assume that $u^N(y) < \phi_1(y) \leq \phi_2(y)$ and from equation 2.5 we get

$$\beta_\epsilon^N(u^N(y) - \phi_1(y)) \geq 0 - C + F(D^2\phi_1(y))$$

That is, $\eta_1 \geq D$ for some D not depending on ϵ, N

Observation 2: $\eta_2 := \beta_\epsilon^N(\phi_2 - u^N)$ is uniformly bounded by above. The proof of this fact is completely analog to the proof of Observation 4.

From the previous two observations we have $|\beta_\epsilon^N(u^N - \phi_1) - \beta_\epsilon^N(\phi_2 - u^N)| \leq D$ where D is a constant that does not depend on ϵ, N.

And hence, from Calderon Zygmund and the Sobolev embeddings we have $\|D^2u^N\|_{C^\alpha(B_1)} \leq C$ where C does not depend on ϵ, N. In particular $u^N, u^N - \phi_1$ and $\phi_2 - u^N$ are uniformly bounded, so if we take N sufficiently big we have

$$F(D^2u^N) = \beta_\epsilon^N(u^N - \phi_1) - \beta_\epsilon^N(\phi_2 - u^N) = \beta_\epsilon(u^N - \phi_1) - \beta_\epsilon(\phi_2 - u^N)$$

And hence $u^N = u^\epsilon$ is a solution of Equation 2.2 □

Theorem 2.1.2. *Let $\phi_1, \phi_2 \in C^2(B_1)$ then the problem 2.1 has a continuous viscosity solution.*

Proof. From Lemma 2.1.1, the Sobolev embeddings and Arsela-Ascoli we know that up to a subsequence of $\epsilon \to 0$ we have $u_\epsilon \to \widehat{u}$ on $C^{1,\alpha}(B_1)$ for some function $\widehat{u} \in C^{1,\alpha}(B_1)$ we want to show that this function $\widehat{u}$ is indeed a solution of our Double obstacle problem (Equation 2.1)

From uniform convergence it follows that $\phi_1 \leq \widehat{u} \leq \phi_2$, to see this we proceed by contradiction. Suppose that $u(x) - \phi_1(x) = -\delta$ for some $x \in B_1$ and $\delta > 0$. Then for $\epsilon > 0$ sufficiently small we have $u_\epsilon(x) - \phi_1(x) < -\frac{\delta}{2}$ but then from the properties of β_ϵ we get

$$\lim_{\epsilon \to 0} \beta_\epsilon(u_\epsilon(x) - \phi_1(x)) \leq \lim_{\epsilon \to 0} \beta_\epsilon(-\frac{\delta}{2}) = -\infty$$

which contradicts Observation 1 and Observation 2 in the proof of Lemma 2.4. If we had $\widehat{u}(x) > \phi_2(x)$ for some $x \in B_1$ we would get a similar contradiction.

Let $x \in B_1$ such that $\phi_1(x) < \widehat{u}(x) < \phi_2(x)$, then, as $\widehat{u} \in C^\alpha$ we know that $\phi_1 < \widehat{u} < \phi_2$ on $B_\delta(x)$ for δ sufficiently small, moreover from uniform convergence we get $\phi_1 < u_\epsilon < \phi_2$ on $B_\delta(x)$ when $\epsilon > 0$ is sufficiently small by redefining δ. And hence, as $\lim_{\epsilon \to 0} \beta_\epsilon(s) = 0$ when $s > 0$ we get $F(D^2\widehat{u}(x)) = 0$.

If $\phi_1(x) = \widehat{u}(x)$ we have (as $\phi_1 < \phi_2$) that $\lim_{\epsilon \to 0} \beta_\epsilon(\phi_2 - u_\epsilon) = 0$ and hence $F(D^2\widehat{u}(x)) \leq 0$. The situation when $\widehat{u}(x) = \phi_2(x)$ follow in the same way □

2.1.2 Regularity

We now study the regularity of a continuous function u that satisfies Equation 2.1. We start with the statement of a classic regularity result, the L^ϵ lemma, that is

Lemma 2.1.3. *Let u nonnegative super solution on B_r, that is $tr(AD^2u) \leq 0$ where $A = A(x)$ is a uniformly elliptic matrix with $\lambda \leq A \leq \Lambda$ then*

$$||u||_{L^\epsilon_w(B_{r/2})} \leq Cu(0)r^n \tag{2.6}$$

or, equivalently

$$|\{u > N\} \cap B_{r/2}| \leq C\frac{u(0)r^n}{N^\epsilon} \quad \textit{for any} \quad N > 0 \tag{2.7}$$

where $\quad C, \epsilon > 0 \quad$ are universal constants

Proof. This is Lemma 4.5. in [6] □

As a motivation for the previous lemma, notice that if a nonnegative function u satisfies a mean value property (for instance when $\Delta u = 0$), Lemma 2.1.3 follows immediately (with $\epsilon = 1$) since

$$u(0) = \frac{1}{|B_{r/2}|}\int_{B_{r/2}} u \geq \frac{1}{|B_{r/2}|}\int_{B_{r/2}\cap\{u\geq N\}} u \geq \frac{N|B_{r/2}\cap\{u\geq N\}|}{|B_{r/2}|} \tag{2.8}$$

Remark 2.1.1. If u is a viscosity solution of the double obstacle problem (Equation 2.1) and γ is a constant such that $\phi_1 \leq \gamma \leq \phi_2$ then $w := \min(u, \gamma)$ is a subsolution of the operator $F(M, x) := tr(A(x)M)$, that is $tr(AD^2 w) \leq 0$ in B_1. Similarly, $\min(u, \gamma)$ is a supersolution of F

The following result will allow us to reduce the problem to that of a single obstacle problem.

Lemma 2.1.4. *Let u a solution of Equation 2.1 with $\phi_1, \phi_2 \in C^\alpha(B_1)$ uniformly separated and let $y \in B_{\frac{1}{2}}$ such that $u(y) = \phi_1(y)$ then there exist a universal radius $r_0 > 0$ (not depending on y) in which u does not touch the upper obstacle, that is*

$$\sup_{B_{r_0}} u < \inf_{B_{r_0}} \phi_2 \tag{2.9}$$

Similarly, if we had that $u(y) = \phi_2(y)$ instead it will follow that

$$\inf_{B_{r_0}} u > \sup_{B_{r_0}} \phi_1 \tag{2.10}$$

Proof. We prove Equation 2.9; the rest of the lemma follows in an analog way. Assume without loss of generality that $\phi_2 - \phi_1 > 1$ and that $y = 0$ is a point in which u touches the lower obstacle. Assume moreover that $u(0) = \phi_1(0) = \sigma(r)$ where $\sigma(r) = Cr^\alpha$ is the modulus of continuity of both obstacles. Arguing by contradiction, lets suppose there is a point $z \in B_{r/2}$ (for some radius r to be chosen) in which u touches the upper obstacle; as the obstacles are uniformly separated we have $u(z) = \phi_2(z) = M > \frac{1}{2}$ and hence $u \leq M + \sigma(r)$ on $B_{r/2}(z)$

Define the constants $\Phi_2 := \inf_{B_r} \phi_2$ and $\Phi_1 := \inf_{B_r} \phi_1$. From Remark 2.1.1 we can apply Lemma 2.1.3 to $\min(u, \Phi_2)$ at 0 and $\max(u, \Phi_1)$ at z to get

$$|\{u > \frac{1}{4}\} \cap B_r| \leq \frac{Cr^n \sigma(r)}{(\frac{1}{4})^\epsilon} = \frac{Cr^{n+\alpha}}{(\frac{1}{4})^\epsilon} \tag{2.11}$$

and

$$\begin{aligned} |\{u \leq \frac{1}{4}\} \cap B_{r/2}(z)| &\leq \frac{C(\frac{r}{2})^n \sigma(r)}{(M + \sigma(r) - \frac{1}{4})^\epsilon} \\ &\leq \frac{\frac{C}{2^n} r^{n+\alpha}}{(M - \frac{1}{4})^\epsilon} \leq \frac{Cr^{n+\alpha}}{(\frac{1}{4})^\epsilon} \end{aligned} \tag{2.12}$$

where C, $\epsilon > 0$ are universal constants. After redefining $C > 0$ if necessary, from the previous two equations we have

$$|B_{r/2}(z)| = C_1 r^n = |\{u > \frac{1}{4}\} \cap B_{r/2}(z)| + |\{u \leq \frac{1}{4}\} \cap B_{r/2}(z)| \leq C_2 r^{n+\alpha}$$

where $C_1, C_2 > 0$ are universal constants and hence $\frac{C_1}{C_2} \leq r^\alpha$, so if we take r small enough we get a contradiction $\square$

From the previous lemma we know that locally around the contact set, our situation is reduced to that of a single obstacle problem with uniformly elliptic coefficients. That is, we know that there exists a universal radius $r_0 > 0$

so that if $y \in B_{1/2}$ and $u(y) = \phi_1(y)$ then u does not touch the upper obstacle in $B_{r_0}(y)$ and hence for $r < r_0$ we have

$$\begin{cases} \phi_1 \le u & \text{in} \quad B_r(y) \\ tr(AD^2u) = 0 & \text{in} \quad \{u > \phi_1\} \cap B_r(y) \\ tr(AD^2u) \le 0 & \text{in} \quad B_r(y) \end{cases} \tag{2.13}$$

Similarly, if we had that $u(y) = \phi_2(y)$ instead we would have the analog set of equations

$$\begin{cases} \phi_2 \ge u & \text{in} \quad B_r(y) \\ tr(AD^2u) = 0 & \text{in} \quad \{u < \phi_2\} \cap B_r(y) \\ tr(AD^2u) \ge 0 & \text{in} \quad B_r(y) \end{cases} \tag{2.14}$$

The following lemmas follow as in [7] and [21] since they only rely on the linearity of the coefficients.

Lemma 2.1.5. *Let $\phi_1 \ge 0$ continuous with modulus of continuity $\sigma(r)$, let u as in Equation 2.13 with $u(y) = \phi_1(y) = \sigma(r)$. Then*

$$\sup_{B_{\frac{r}{2}}(y)} u \le C\sigma(r) \tag{2.15}$$

Similarly, if $\phi_2 \le 0$ has modulus of continuity $\sigma(r)$ and u satisfies Equation 2.14 with $u(y) = \phi_2(y) = -\sigma(r)$ then

$$\inf_{B_{\frac{r}{2}}(y)} u \ge -C\sigma(r) \tag{2.16}$$

For some universal constant $C > 0$.

Proof. We prove Equation 2.15, the rest of the lemma follows in a similar way. Assume without loss of generality that $y = 0$. Consider v, w continuous on B_r satisfying

$$\begin{cases} tr(AD^2w) = & 0 \quad \text{in} \quad B_r(y) \\ w = & u \quad \text{on} \quad \partial B_r(y) \end{cases} \tag{2.17}$$

$$\begin{cases} tr(AD^2v) = & tr(AD^2u) \quad \text{in} \quad B_r(y) \\ u = & 0 \quad \text{on} \quad \partial B_r(y) \end{cases} \tag{2.18}$$

From Equation 2.17, the superharmonicity of u and Harnack inequality we get

$$\sup_{B_{r/2}} w \leq C \inf_{B_{r/2}} w \leq Cw(0) \leq Cu(0) = C\sigma(r)$$

Notice from Equation 2.18 that v can only attain its maximum in the interior of B_r, moreover, this maximum can only be attained in a point z in which $tr(D^2u(z)) < 0$ and this only happens when $u(z) = \phi_1(z)$ and as $w \geq 0$ we get

$$\sup_{B_{r/2}} v = v(z) = u(z) - w(z) \leq u(z) = \phi(z) \leq 2\sigma(r)$$

As $u = v + w$ the lemma follows □

Lemma 2.1.6. *If u is a solution of Equation 2.1 with Hölder continuous and uniformly separated obstacles then $u \in C^\alpha(B_{1/2})$. Moreover, $||u||_{C^\alpha(B_{1/2})} \leq C$ for some positive constant C that depends on $\phi_1, \phi_2, n, \lambda, \Lambda$*

Proof. This is proof is just a natural modification of the Hölder continuity result when there is only one obstacle that can be found in [7].

Assume $\phi_2 - \phi_1 > 1$. And define $K_1 := \{u = \phi_1\}$ and $K_2 = \{u = \phi_2\}$ and $K := C_1 \cup C_2$. Notice that from Lemma 2.1.4 we know that K_1 and K_2

are uniformly separated, that is $d(K_1, K_2) > \frac{r_0}{2}$. Moreover, if we assume that both ϕ_1 and ϕ_2 have modulus of continuity $\sigma(r) = Cr^\alpha$, from Equation 2.15 (or Equation 2.16 depending on the case) we have

$$|u(x) - u(x')| \leq C|x - x'|^\alpha \quad \text{for} \;\; x, x' \in B_{1/2} \;\; \text{and} \;\; x' \in K \tag{2.19}$$

Let $x, y \in B_{\frac{1}{2}}$ and let $x', y' \in K$ such that $|x - x'| = d(x, K)$ and $|y - y'| = d(y, K)$, assume without loss of generality that $|x, x'| \leq |y, y'|$. Moreover, let $z := \frac{x+y}{2}$, $r := d(z, K) = |z - z'|$, $d := |x - y|$. Consider the following cases

Case 1: If $|x - y| > \frac{r_0}{4}$ we have

$$|u(x) - u(y)| \leq 2(||\phi_1||_{L^\infty} + ||\phi_2||_{L^\infty}) \tag{2.20}$$

Case 2: If $|x - y| < |z - z'|$ we have that $x, y \in B_{r/2}(z)$ and $H(D^2u) = tr(AD^2u) = 0$ on $B_r(z)$ where A is a uniformly elliptic matrix as on Remark 2.33 and hence we can use the Hölder estimates for elliptic equations to get

$$||u - u(z)||_{C^\alpha(B_r(z))} \leq \frac{||u - u(z)||_{L^\infty(B_r(z)))}}{r^\alpha}$$

And from Equation 2.19 we conclude

$$|u(x) - u(y)| \leq C|x - y|^\alpha \tag{2.21}$$

Case 3: If $|x - y| \geq |z - z'|$. In this case, there are two possibilities. If $|x - y| > \frac{r_0}{4}$ Case 1 will apply, otherwise from Lemma 2.1.4 we can guarantee that either $x', y' \in K_1$ (or $x', y' \in K_2$) and hence

$$\begin{aligned}
|u(x) - u(y)| &\leq |u(x) - u(z')| + |u(z') - u(y)| \\
&\leq C|x - z'|^\alpha + |y - z'|^\alpha \\
&\leq \hat{C}(|x - z'| + |y - z'|)^\alpha \qquad (2.22) \\
&= \hat{C}(|x - z| + |y - z| + 2|z - z'|)^\alpha \\
&\leq 2\hat{C}(|x - y|)^\alpha
\end{aligned}$$

Where we are using that the function $f(w) = w^\alpha$ is concave for $w > 0$, so $\frac{w_1^\alpha + w_2^\alpha}{2} \leq \left(\frac{w_1 + w_2}{2}\right)^\alpha$.

From Equation 2.20, Equation 2.21 and Equation 2.22 it follows that $u \in C^\alpha(B_{1/2})$ as desired □

Lemma 2.1.7. *Let $\phi_1 \in C^1(B_r)$, let $\sigma(r)$ the modulus of continuity of $\partial_e \phi_1$ (in any direction e). Let u as in Equation 2.13 with $u(y) = \phi_1(y)$. Then*

$$\sup_{B_{\frac{r}{2}}(y)} (u - L) \leq Cr\sigma(r) \qquad (2.23)$$

Where L is the tangent line of ϕ_1 at y, that is $L(x) := \phi_1(y) + \nabla\phi_1(y) \cdot x$ and $C > 0$ is a positive constant that does not depend on u.

Similarly, if $\phi_2 \in C^1(B_r)$, $\sigma(r)$ is the modulus of continuity of $\partial_e \phi_2$ and u satisfies Equation 2.14 with $u(y) = \phi_2(y)$ we have that

$$\inf_{B_{\frac{r}{2}}(y)} (u - L) \geq -Cr\sigma(r) \qquad (2.24)$$

where L is the tangent line of ϕ_2 at y.

Proof. Assume without loss of generality that $y = 0$. Let $w := u - L + r\sigma(r)$; as w solves the obstacle problem of Equation 2.13 with obstacle $\phi_1 - L$ we can apply Lemma 2.1.5 to w and get Equation 2.24 since $\phi_1 - L$ has modulus of continuity $r\sigma(r)$ $\square$

Theorem 2.1.8. *Let u a solution of Equation 2.1 where the obstacles ϕ_1 and ϕ_2 are smooth and uniformly separated and $A(x) \in C^\alpha(B_1)$ then $u \in C^{1,1}$*

Proof. We assume without loss of generality that $\phi_2 - \phi_1 > 1$ and let $\sigma(r) = Cr$ de modulus of continuity of $\partial_e \phi_1$ and $\partial_1 \phi_2$ in any direction e.

Take $y \in \{\phi_1 < u < \phi_2\} \cap B_{\frac{1}{2}}$, and let r_0 as in Lemma 2.1.4; starting from $r = 0$, start growing the radius of a ball $B_r(y)$ until either $r = \frac{r_0}{2}$ or $\partial B_r(y)$ touches the free boundary (whichever comes first)

In the first scenario (when $r = r_0/2$) we have that $H(D^2u, x) := tr(AD^2u) = 0$ in $B_r(y)$ with $A(x) \in C^\alpha$ and hence we can use the Schauder estimates for linear equations applied to the operator H to get

$$|D^2u(y)| \leq \frac{C||u||_{L^\infty}}{r_0^2} \tag{2.25}$$

In the second scenario we assume that $\partial B_r(y)$ touches the free boundary of the lower obstacle at x' and hence $\phi_1(x') = u(x')$, let L the tangent plane of ϕ_1 at x', so from Lemma 2.1.7 we have that $(u - L) \leq 2Cr^2$ in $B_r(y)$, and we can apply Schauder estimates to $u - L$ in $B_r(y)$ to get

$$|D^2u(y)| = |D^2u(y) - L(y)| \leq \frac{C||u - L||_{L^\infty(B_r(y))}}{r^2} \leq C \tag{2.26}$$

The result follows from Equation 2.25 and 2.26 $\square$

We want to point out that because of the local uniform bound of u found in Lemma 2.1.4, the study of the free boundary of the double obstacle problem in Equation 2.1 reduces to the study of the analog single obstacle problem. This latter problem was addressed by Caffarelli in Section 8 of [3] where he showed under an extra density assumption in the contact set that the free boundary is a $C^{1,\alpha}$ manifold.

In the following section we study a situation in which the free boundary presents an extra challenge that we will have to address.

2.2 The fully nonlinear elliptic case

In the previous section we studied the regularity of the solution of a problem involving two obstacles and linear elliptic operators. We now move to a situation that involves fully nonlinear elliptic operators and was motivated by a result of Ki-Ahm Lee in [21]. He studied the regularity of the free boundary of a single obstacle problem in which the involved operator was elliptic and concave. More precisely he studied u satisfying

$$\begin{cases} \phi \le u & \text{in} \quad B_1 \\ F(D^2u) \le 0 & \text{in} \quad B_1 \\ F(D^2u) = 0 & \text{in} \quad \{u > \phi\} \cap B_1 \end{cases} \tag{2.27}$$

where ϕ is a smooth obstacle and F is a convex fully nonlinear elliptic operator. Ki-Ahm Lee showed that in this situation the free boundary is still a $C^{1,\alpha}$ manifold under the same conditions on the contact set stated in the item (3) of Chapter 1 . With this in mind we wanted to study the free boundary of a double obstacle problem that locally near the obstacles satisfied conditions similar to those studied by Ki-Ahm Lee. To introduce this problem we first define the positive Pucci operator

$$M^{+}(D^2u) := \sup_{\lambda I \leq A \leq \Lambda I} tr(AD^2u), \tag{2.28}$$

the negative Pucci operator

$$M^{-}(D^2u) := \inf_{\lambda I \leq A \leq \Lambda I} tr(AD^2u) \tag{2.29}$$

And

$$H(D^2u, u) := \frac{u-\phi_1}{\phi_2-\phi_1} M^{-}(D^2u) + \frac{\phi_2-u}{\phi_2-\phi_1} M^{+}(D^2u) \tag{2.30}$$

Let $\phi_1 < \phi_2$ smooth obstacles that are uniformly separated in B_1, and let $g : \mathbb{R} \to \partial B_1$ a smooth function compatible with ϕ_1 and ϕ_2 in the sense that $\phi_1 < g < \phi_2$ in ∂B_1. In this section we study a function u that satisfies

$$\begin{cases} \phi_1 \leq u \leq \phi_2 & \text{in} \quad B_1 \\ u = g & \text{on} \quad \partial B_1 \\ H(D^2u, u) \leq 0 & \text{in} \quad \{u < \phi_2\} \cap B_1 \\ H(D^2u, u) \geq 0 & \text{in} \quad \{u > \phi_1\} \cap B_1 \end{cases} \tag{2.31}$$

This problem has the following feature: exactly in the middle of the obstacles, the function u transitions from solving a convex operator to solving a concave operator. More precisely, if we take $y \in B_1$ with $u(y) = \frac{\phi_1(y)+\phi_2(y)}{2}$ we have that $H(D^2u, u) = \frac{\Lambda+\lambda}{2}\Delta u$ at y. Similarly, if $u(y) < \frac{\phi_1(y)+\phi_2(y)}{2}$ we have that $H(D^2u, u)$ is a convex operator at y and if $u(y) > \frac{\phi_1(y)+\phi_2(y)}{2}$ we have that $H(D^2u, u)$ is a concave operator at y.

It is natural to ask then if the boundary of the set in which H is convex is indeed smooth. In the rest of the section we will answer this question and will study the regularity of u and its free boundary (i.e. $\partial\{\phi_1 < u < \phi_2\}$)

Lemma 2.2.1. *Let u such that $H(D^2u, u) = 0$ in B_1 with H defined as in Equation 2.30. Then u is a solution of a linear elliptic equation of the form $tr(AD^2u) = 0$ in B_1 for some elliptic matrix $A = A(x)$ with $\lambda \leq A \leq \Lambda$*

Proof. Fix $x \in B_1$ and define

$$\alpha = \alpha(x) := \frac{u(x) - \phi_1(x)}{\phi_2(x) - \phi_1(x)} \tag{2.32}$$

After a change of coordinates we can write $D^2u(x)$ as a diagonal matrix whose positive eigenvalues are $\{e_i\}_{i=1}^n$ and whose negative eigenvalues are $\{\hat{e}_i\}_{i=1}^m$ and hence

$$\begin{aligned} H(D^2u, u) &= \alpha M^-(D^2u) + (1-\alpha)M^+(D^2u) \\ &= \alpha\Big(\Lambda\sum_{i=1}^m \hat{e}_i + \lambda\sum_{i=1}^n e_i\Big) + (1-\alpha)\Big(\Lambda\sum_{i=1}^n e_i + \lambda\sum_{i=1}^m \hat{e}_i\Big) \\ &= \Big(\alpha(\Lambda-\lambda) + \lambda\Big)\sum_{i=1}^m \hat{e}_i + \Big(\Lambda - \alpha(\Lambda-\lambda)\Big)\sum_{i=1}^n e_i \end{aligned} \tag{2.33}$$

Notice that as $0 \leq \alpha \leq 1$, from Equation 2.33 we have $H(D^2u, u) = tr(AD^2u)$ at x, where $A = A(x)$ is a uniformly elliptic matrix with $\lambda \leq A(x) \leq \Lambda$ as desired. $\square$

Lemma 2.2.2. *If u is a solution of Equation 2.31 then u is Hölder continuous in the interior of B_1. Moreover, there exists a constant $C > 0$ that does not depend on u such that $||u||_{C^\alpha(B_{1/2})} \leq C$*

Proof. From Lemma 2.2.1 we know that there exists a matrix $A = A(x)$ with $\lambda \leq A \leq \Lambda$, such that u satisfies $tr(AD^2u) = 0$ in $\{\phi_1 < u < \phi_2\}$, moreover, u satisfies

$$\begin{cases} \phi_1 \leq u \leq \phi_2 & \text{in} \quad B_1 \\ u = g & \text{on} \quad \partial B_1 \\ tr(AD^2u) \leq 0 & \text{in} \quad \{u < \phi_2\} \cap B_1 \\ tr(AD^2u) \geq 0 & \text{in} \quad \{u > \phi_1\} \cap B_1 \end{cases} \tag{2.34}$$

and hence from Lemma 2.1.6 we know that $u \in C^\alpha(B_{1/2})$ □

At this moment we can see that in between the obstacles u is the solution of an elliptic operator that is a perturbation of the Laplacian in the sense of the following

Lemma 2.2.3. *Let u a solution of Equation 2.31, let $y \in B_{1/2}$ and $\delta > 0$ such that $H(D^2u, u) = 0$ in $B_\delta(y)$ There exist constants $\bar{\lambda}, C, \alpha > 0$ and a matrix $\bar{B} = \bar{B}(x)$ such that $tr\big((\bar{\lambda}I + \bar{B})D^2u\big) = 0$ in $B_\delta(y)$ and $|\bar{B}(x)| \leq C|x - y|^\alpha$ where $\bar{\lambda}, C, \alpha > 0$ do not depend on u or y.*

Proof. Take $x \in B_\delta(y)$ and change coordinates so that $D^2u(x)$ is a diagonal matrix with positive eigenvalues $\{e_i\}_{i=1}^n$ and negative eigenvalues $\{\hat{e}_i\}_{i=1}^m$, moreover, assume without loss of generality that $u(y) = 0$. At x we have

$$\begin{aligned} H(D^2u, u) &= \frac{u - \phi_1}{\phi_2 - \phi_1} M^-(D^2u) + \frac{\phi_2 - u}{\phi_2 - \phi_1} M^+(D^2u) \\ &= \frac{u}{\phi_2 - \phi_1}\big(M^-(D^2u) - M^+(D^2u)\big) + \big(M^-(D^2u) + M^+(D^2u)\big) \\ &= \frac{u}{\phi_2 - \phi_1}\Big((\lambda - \Lambda)\sum_{i=1}^n e_i + (\Lambda - \lambda)\sum_{i=1}^m \hat{e}_i\Big) + (\lambda + \Lambda)tr(ID^2u) \\ &= \frac{u}{\phi_2 - \phi_1} tr(BD^2u) + (\lambda + \Lambda)tr(ID^2u) \end{aligned} \tag{2.35}$$

For some matrix $B = B(x)$ with $B_{ii} \in \{(\Lambda - \lambda), (\lambda - \Lambda)\}$, and hence, at x we have

$$H(D^2u, u) = tr\left(\left((\lambda + \Lambda)I + \frac{uB}{\phi_2 - \phi_1}\right)D^2u\right) = 0 \tag{2.36}$$

From Lemma 2.2.2 we know $u \in C^\alpha$, our assumption that $u(y) = 0$ and the uniform separation of our obstacles (say $\phi_2 - \phi_1 > 1$ in B_1) we have

$$\left|\frac{u(x)}{\phi_2(x) - \phi_1(x)} - \frac{u(y)}{\phi_2(y) - \phi_1(y)}\right| = \frac{|u(x) - u(y)|}{|\phi_2(x) - \phi_1(x)|} \leq C|x - y|^\alpha \tag{2.37}$$

And hence $H(D^2u, u) = tr\left(\bar{\lambda}I + \overline{B}D^2u\right) = 0$ at x. Where we defined $\bar{\lambda} := (\lambda + \Lambda)$ and $\overline{B} = \overline{B(x)} := \frac{uB}{\phi_2 - \phi_1}$. Notice moreover from Equation 2.37 that $|B(x)| \leq C|x - y|^\alpha$ as desired. □

At this moment we can solve one of the questions that motivated this section that was stated in the paragraph after Equation 2.31, that is

Lemma 2.2.4. *Let u a solution of Equation 2.31, let $A := \{u(x) < \frac{\phi_1(x) + \phi_2(x)}{2}\}$ then $B_{1/2} \cap \partial A$ is locally a $C^{1,\alpha}$ manifold around regular points, that is, around points in which $|Du(x)| > 0$.*

Proof. Take a $x \in \partial A$, so that $u(x) = \frac{\phi_1(x) + \phi_2(x)}{2}$ from Lemma 2.2.3 we know that there is a small neighborhood $B_\delta(x)$ where u satisfies an equation of the form

$$tr\left((\bar{\lambda}I + \bar{B})D^2u\right) = 0$$

for a matrix B that grows away from x in a Hölder way. That is, u is the solution of a small perturbation of the Laplace operator and hence from the Schauder estimates (see Theorem 2 in [4]) we know that $u \in C^{2,\alpha}(B_{\delta/2})(x)$ and hence the level sets of u are $C^{2,\alpha}$ at regular points (i.e. at points in which $|Du| \neq 0$)

□

The following is an immediate consequence of the fact that u solves a double obstacle problem with linear coefficients as the one studied in (see Equation 2.34) and will be improved using results from [21]

Lemma 2.2.5. *Let u a solution of Equation 2.31 where the obstacles ϕ_1 and ϕ_2 are smooth and uniformly separated then $u \in C^{1,\alpha}(B_{\frac{1}{2}})$, moreover $||u||_{C^{1,\alpha}(B_{1/2})} \leq C$ where $C > 0$ is a universal constant that does not depend on u.*

Proof. The proof of this lemma is exactly as that of Theorem 2.1.8. Notice that in order to prove Theorem 2.1.8 we need two ingredients:

The first one is the fact that u grows away from the obstacles in a quadratic way, that is we need Lemma 2.1.7 but this lemma also applies in our situation.

The second ingredient is the Schauder estimates for second order linear equations. In our situation we can use the Schauder estimates for fully nonlinear elliptic equations of [6] instead and repeat the same proof. □

Theorem 2.2.6. *Let u a solution of Equation 2.31 where the obstacles ϕ_1 and ϕ_2 are smooth and uniformly separated then $u \in C^{1,1}(B_{1/2})$ and the free* *boundary is a $C^{1,\alpha}$ manifold near regular points.*

Proof. Notice from Lemma 2.2.2 that the solution of this problem is locally uniformly bounded near the obstacles, and hence locally near the obstacles our problem satisfies the hypothesis of [21] and hence locally, our function is $C^{1,1}$ and the free boundary (i.e.$\partial(B_{1/2} \cap \{\phi_1 < u < \phi_2\})$) is locally a $C^{1,\alpha}$ manifold. If we are away from the obstacle then Lemma 2.2.3 applies. □

2.3 The parabolic case

We now study a game that is a slight modification of the one described in Section 2.1. Suppose one more time that we have a particle that moves inside B_1 at each instant of time in the following way: if the particle is in the position x at time $t = t_0$ then, the location of the particle at $t = t_0 + \Delta_t$ will be given by a uniform distribution in the region $E_{A(x)}(x)$. As before, $E_A(y)$ denotes an ellipsoid with ellipticity A centered at y.

Suppose now that there is a 'player' that is playing a game against 'the house' with the following rules:

1. There is a *payoff* function $\phi_1(x)$ (with $\phi_1 : B_1 \to R$) and the player can stop the game at any instant of time and leave with an amount of money given by $\phi_1(x)$. That is, if in a certain instant of time the particle is located at the position x then the player can choose to leave the game with a profit of $\phi_1(x)$.

2. The house can stop the game at any moment. If this is the case, the player will leave with an amount of money given by $\phi_2(x)$ (with $\phi_2 : B_1 \to R$ and $\phi_2 > \phi_1$) where x denotes the position of the particle at that instant of time.

3. If the particle touches the border of the ball or if it has passed a certain amount of time since the beginning of the game (say at time $t = 1$), the game is forced to stop and the player leaves with a prescribed amount of money given by $\hat{g}(x,t)$. In other words, there is a function $\hat{g} : \hat{\partial}_p Q_1 \to R$ and if the particle is located at x in time t with $(x,t) \in \hat{\partial}_p Q_1$ the game is forced to stop and the player leaves with an amount of money given by $\hat{g}(x,t)$. Where $\hat{\partial}_p Q_1 := (\partial B_1 \times [0,1]) \cup (B_1 \times \{1\})$

In this scenario, let $v(x,t)$ the expected value that the player obtains from this game if the game started at time t with the particle located at x and define. Let $u(x,t) := v(x,1-t)$, then u satisfies the following equations

$$\begin{cases} \phi_1 \le u \le \phi_2 & \text{in} \quad Q_1 \\ u = g & \text{on} \quad \partial_p Q_1 \\ F(D^2u, x) - \partial_t u \le 0 & \text{in} \quad \{u < \phi_2\} \cap Q_1 \\ F(D^2u, x) - \partial_t u \ge 0 & \text{in} \quad \{u > \phi_1\} \cap Q_1 \end{cases} \tag{2.38}$$

Where $g(x,t) := \hat{g}(x,1-t)$ and $\partial_p Q_1$ denotes the parabolic boundary of Q_1 (i.e. $\partial_p Q_1 := (\partial B_1 \times [0,1]) \cup (B_1 \times \{0\})$) and F is the linear operator $F(M,x) = tr(A(x)M)$ for a uniformly elliptic matrix $A = A(x)$ with ellipticity constants λ, Λ. For simplicity we consider $\phi_1, \phi_2 \in C^2(\overline{Q_1})$ such that $\phi_1 < \phi_2$.

If u solves Equation 2.38 we say that u solves the parabolic double obstacle problem (ϕ_1, ϕ_2, F, Q_1)

Remark 2.3.1. If u is a viscosity solution of the parabolic double obstacle problem $(\phi_1, \phi_2,\ F,\ Q_1)$ and γ is a constant such that $\phi_1 < \gamma < \phi_2$ then $w := \max(u, \gamma)$ is a subsolution of F, that is $F(D^2 w) - \partial_t w \geq 0$ on B_1. Similarly, $\min(u, \gamma)$ is a supersolution of F

The following is the analog of Lemma 2.1.3 to the parabolic situation

Lemma 2.3.1. *(parabolic-L^{ϵ}_{w}) Let $w \in C(Q_1)$ a nonnegative supersolution of the operator $F - \partial_t$ (i.e. $F(D^2 w) - \partial_t w \leq 0$), $0 < R < 1$ and $-1 < t_0 < 0$ then there exist universal a universal constant $C > 0$ such that*

$$||w||_{L^{\epsilon}(\widehat{K}_1)} \leq C w(0,0)$$

that is,

$$|\{w > N\} \cap \widehat{K}_1| \leq \left(\frac{Cw(0,0)}{N}\right)^{\epsilon} \quad \textit{for any} \quad N > 0 \tag{2.39}$$

where $\widehat{K}_1 := B_R \times (-1, t_0)$

Proof. See Theorem 4.5. on [15]. □

In order to motivate the previous lemma, consider the heat operator in one spatial dimension, that is, let w satisfying

$$\begin{cases} w'' - \partial_t w = 0 \quad \text{on} \ \ Q_1 \\ \qquad w \geq 0 \ \ \text{on} \ \ Q_1 \end{cases}$$

From Theorem 3 section 2.3. in Evans book ([10]) we know that w satisfies a mean value formula, let $E = E(0,0,1)$ the heat ball centered at $(0,0)$ with radius 1. Then

$$w(0,0) = \frac{1}{4(2)^n} \iint_E w(x,t)\frac{|x|^2}{t^2} dxdt \geq \frac{1}{4(2)^n} \iint_{E\cap\{w\geq N\}} w(x,t)\frac{|x|^2}{t^2} dxdt$$

Notice that for any $A \subset Q_1$ we have $\iint_A |x|^2 dxdt \geq \frac{|A|^3}{4}$, so it follows that

$$w(0,0) \geq \frac{N}{4(2)^n} \iint_{E\cap\{w\geq N\}} |x|^2 dxdt \geq CN|E \cap \{w \geq N\}|^3$$

And hence Equation 2.39 holds in this particular case if we pick $\widehat{K}_1 \subset E$.

Lemma 2.3.2. *Let $w : Q_1^- \to \mathbb{R}$ a nonnegative function such that $F(D^2w) - \partial_t w = 0$ and $w(x_0,t_0) > 0$ for some $t_0 < 0$ then $w(0,0) > 0$*

Proof. This follows by contradiction using Lemma 2.3.1 □

Lemma 2.3.3. *(Hölder growth at contact points)Let $\sigma(r) = Ar^\alpha$ be the modulus of continuity of ϕ_1 and ϕ_2. u solves the problem (ϕ_1, ϕ_2, F, Q_2) with $\lambda \leq F \leq \Lambda$. Let $X_0 \in E \cap Q_{1/2}$ a contact point. Then u grows in a $C^\beta(Q_{1/2})$ away from X_0. That is, there exist a universal constant A such that*

$$|u(X) - u(X_0)| \leq Ar^\beta \text{ for } X \in Q_r(X_0) \text{ and } 0 < r < 1/2$$

Proof. We consider the situation in which $X_0 \in E_1 \cap Q_{1/2}$, the other case is analog. We argue by contradiction: if this result did not hold we would have two sequences of positive real numbers $\{A_i\}_i$ and $\{r_i\}_i$ satisfying $\lim_{i\to\infty} A_i = \infty$ and $\lim_{i\to\infty} r_i = 0$ and for each $i \in \mathbb{N}$ we would have:

$$\begin{cases} \text{Two obstacles } \widehat{\phi_1^i} < \widehat{\phi_2^i} \text{ on } Q_2 \text{ with modulus of continuity } \sigma \\ \text{A solution } \widehat{u}_i \text{ of the problem } (\widehat{\phi_1^i}, \widehat{\phi_2^i}, F, Q_2) \text{ with } \lambda \le F \le \Lambda \\ \text{A point } Z_i \in Q_{1/2} \text{ so that } \widehat{\phi_1^i}(Z_i) = \widehat{u}_i(Z_i) \\ |\widehat{u}_i(X) - \widehat{u}_i(Z_i)| \le A_i r_i^\beta \text{ for } X \in Q_{r_i}(Z_i) \\ \text{A point } X_i \in Q_{\delta r_i}(Z_i) \text{ so that } |\widehat{u}_i(\widehat{X}_i) - \widehat{u}_i(Z_i)| > A_i(\delta r_i)^\beta \end{cases}$$

Where $\delta, \beta > 0$ are to be chosen. We assume that $Z_i = (0,0)$ and define:

$$u_i(X) := \frac{\widehat{u}_i(r_i x, r_i^2 t)}{A_i r_i^\beta} \quad \text{for} \quad X = (x,t) \in Q_1 \tag{2.40}$$

$$\phi_1^i(X) := \frac{\widehat{\phi_1^i}(r_i x, r_i^2 t)}{A_i r_i^\beta} \quad \text{for} \quad X = (x,t) \in Q_1 \tag{2.41}$$

$$\phi_2^i(X) := \frac{\widehat{\phi_2^i}(r_i x, r_i^2 t)}{A_i r_i^\beta} \quad \text{for} \quad X = (x,t) \in Q_1 \tag{2.42}$$

Notice for instance that if we take $\beta < \alpha$ we will get

$$|\phi_1^i(x) - \phi_1^i(0)| \le \frac{\sigma(r_i)}{A_i r_i^\beta} = \frac{A r_i^{\alpha-\beta}}{A_i} \to 0 \quad \text{on} \quad Q_1 \tag{2.43}$$

Moreover, up to a subsequence we have the following:

1. u_i solves a parabolic double obstacle problem $(\phi_1, \phi_2, F_i, Q_1)$ for the rescaled elliptic operator $\lambda \le F_i \le \Lambda$

2. $||\phi_1^i - \phi_1^i(0)||_{L^\infty(B_1)}$, $||\phi_2^i - \phi_2^i(0)||_{L^\infty(B_1)} \leq \frac{1}{i}$ (see Equation 2.43)

3. $|u_i| \leq 1$ on Q_1

4. $u_i(X_i) > \delta^\beta$ for some $X_i \in Q_\delta$

5. $u_i(0) = \phi_1^i(0) = 0$

One of the following situations will occur:

Case 1: There is a subsequence of $\{u_i\}_i$ such that each u_i touches both obstacles on $Q_{2\delta}$.

Let $Y_i := (y_i, t_i)$ and $\hat{Y}_i := (\hat{y}_i, \hat{t}_i)$ in $Q_{2\delta}$ such that $\phi_1^i(Y_i) = u_i(Y_i)$ and $\phi_2^i(Y_i) = u_i(Y_i)$. Notice that $w := u_i - u_i(Y_i) + \frac{1}{i} \geq 0$ on Q_1 so we can apply Lemma 2.3.1 and Remark 2.3.1 to w at Y_i and get

$$|\{u_i > M\} \cap A_1| \leq \left(\frac{Cw(Y_i)}{M}\right)^\epsilon \leq \left(\frac{C}{iM}\right)^\epsilon \tag{2.44}$$

Where $A_1 := B_{\frac{1}{4}}(y_i) \times (t_i - \frac{1}{4}, t_i - \frac{1}{8})$

Notice also that $u_i \leq u_i(\hat{Y}_i) + \frac{1}{i}$ on Q_1 so we can apply apply Lemma 2.3.1 to $h := u_i(\hat{Y}_i) + \frac{1}{i} - u_i$ (see Remark 2.3.1) at $\hat{Y}_i$ and get

$$
\begin{aligned}
|\{u_i \le M\} \cap A_2| &\le \left(\frac{Ch(\hat{Y}_i)}{u_i(\hat{Y}_i) + \frac{1}{i} - M}\right)^{\epsilon} \\
&\le \left(\frac{C}{i(u_i(\hat{Y}_i) + \frac{1}{i} - M)}\right)^{\epsilon} \\
&\le \left(\frac{C}{i(1-M)}\right)^{\epsilon}
\end{aligned} \tag{2.45}
$$

Where $A_2 := B_{\frac{1}{4}}(\hat{y}_i) \times (\hat{t}_i - \frac{1}{4}, \hat{t}_i - \frac{1}{8})$

Notice that if we pick $\delta > 0$ small enough we would have that $|A_1 \cap A_2| \ge c > 0$ for some universal constant c and hence

$$
0 < c \le |A_1 \cap A_2| = \{u_i > M\} \cap (A_1 \cap A_2)| + \{u_i \le M\} \cap (A_1 \cap A_2)|
$$

and if we pick $0 < M < \frac{\delta^{\beta}}{2}$ on Equation 2.44 and Equation 2.45, use the last equation and pass to the limit when $i \to \infty$ we get

$$
0 < c \le \left(\frac{C}{iM}\right)^{\epsilon} + \left(\frac{C}{i(1-M)}\right)^{\epsilon} \to 0
$$

which is a contradiction.

Case 2: there is a subsequence of $\{u_i\}_i$ of functions that does not touch the upper obstacle on $Q_{2\delta}$.

In this situation we follow [24] to get a contradiction. Define $v_i, \widehat{v}_i : Q_1 \to \mathbb{R}$ such that

$$
\begin{cases} F(D^2 v_i) - \partial_t v_i = 0 & \text{on} \quad Q_{2\delta} \\ v_i = u_i & \text{on} \quad \partial_p Q_{2\delta} \end{cases}
$$

$$\begin{cases} F(D^2\widehat{v}_i) - \partial_t \widehat{v}_i = 0 \quad \text{on} \quad Q_{2\delta} \\ \widehat{v}_i = \max(u_i, \frac{1}{i}) \quad \text{on} \quad \partial_p Q_{2\delta} \end{cases}$$

From the comparison principle and as u_i is the smallest super solution above ϕ_1^i we actually have $-\frac{1}{i} \leq v_i \leq u_i \leq \widehat{v}_i$ on $Q_{2\delta}$. From the interior estimates for fully nonlinear parabolic equations (see [27]) and Arsela-Acoli it follows that $v_i \to v_0$ uniformly on Q_δ. We also know that $F(D^2 v_0) - \partial_t v_0 \leq 0$ on Q_δ and as $-\frac{1}{i} \leq v_i(0) \leq u(0) = \phi_1^i(0) = 0$ we have $v_0(0) = 0$ and hence, as $v_0 \geq 0$ we conclude (using Lemma 2.3.1) that $v_0 = 0$ on Q_δ^- .

Notice also that from the interior estimates for parabolic equations and from Arsela-Ascoli we have $\hat{v}_i \to \hat{v}_0$ uniformly on Q_δ and as $0 \leq (\widehat{v}_i - v_i) \leq \frac{2}{i}$ on $\partial_p Q_{2\delta}$ (from the maximum principle) we know $0 \leq (\widehat{v}_i - v_i) \leq \frac{2}{i}$ on $Q_{2\delta}$ and hence $\widehat{v}_0 = u_0 = 0$ on Q_δ^- and, as $u_i \leq \hat{v}_i$

At this point we know that v_0 is caloric, $v_0 \leq 1$ on $Q_{2\delta}$ and $v_0 = 0$ on Q_δ so we can pick $C, d, e > 0$ properly so that the barrier $\psi := C|x|^2 + dt + e$ on Q_δ^+ is supercaloric, $\psi \geq 1$ on $\partial B_\delta \times (0, \delta^2)$ and $\phi(0) = 0$ and hence if we redefine $\delta, \beta > 0$ properly this contradicts the fact that $u_i(X_i) > \delta^\beta$ for some $X_i \in Q_\delta$ □

We now bring the previous estimates to the interior of Q_1.

Theorem 2.3.4. *(C^α regularity) Let u a solution of the parabolic double obstacle problem (ϕ_1, ϕ_2, F, Q_1^-) with $\phi_1, \phi_2 \in C^\alpha$. Then u is Hölder continuous in $Q_{1-\delta}^-$ for any $0 < \delta < 1$*

Proof. This proof follows exactly as Theorem 2.1.6 but using Lemma 2.3.3 to control the growth of the u near the free boundary, the parabolic maximum principle and the parabolic interior estimates (see [27]) instead of its elliptic counterparts $\square$

Lemma 2.3.5. *Let u a solution of the parabolic double obstacle problem (ϕ_1, ϕ_2, F, Q_1^-), suppose $\partial_e\phi_1$ and $\partial_e\phi_2$ have modulus of continuity $\sigma(r) = Ar^\alpha$ on any direction e on space-time, and let X_0 a contact point of the lower obstacle (that is $X_0 \in E_1 \cap Q_{1/2}^-$) then*

$$\sup_{Q_{r/2}(x_0)} (u - \phi_1) \leq Cr^{1+\alpha} \tag{2.46}$$

Similarly, if $x_0 \in E_2 \cap Q_{1/2}$ then

$$\sup_{B_{r/2}(x_0)} (\phi_2 - u) \leq Cr^{1+\alpha} \tag{2.47}$$

Proof. We know from that if $\partial_e\phi_1$ has modulus of continuity σ then

$$|\phi(B) - \phi(A) - (B - A) \cdot \nabla\phi(A)| \leq |B - A|\sigma(|B - A|)$$

Let L_x the first order Taylor expansion of ϕ_1 at x, that is

$$L_x(y) := \phi_1(x) + (y - x) \cdot \nabla\phi_1(x)$$

We now want to show that if $x \in E_1 \cap Q_{1/2}$ then

$$U_x(y) := u(y) - L_x(y) \leq A|y-x|\sigma(|y-x|) \leq A|y-x|^{1+\alpha} \quad \text{for} \quad y \in Q_1(x)$$

For a universal constant $A > 0$.

Notice that U_x solves the parabolic double obstacle problem $(\phi_1 - L_x, \phi_2 - L_x, F, Q_1^-)$. We point out that as the obstacles are uniformly separated and using the previous theorem we know that locally (i.e. around each point on $E_1 \cap Q_{1/2}^-$) our situation reduces to that of a single obstacle problem (with obstacle $\phi_1 - L_x$), in particular, we can apply Lemma 2.3.3 to U_x and conclude the result □

Theorem 2.3.6. *($C^{1,\alpha}$ regularity) Let u a solution of the parabolic double obstacle problem (ϕ_1, ϕ_2, F, Q_1^-) with $\phi_1, \phi_2 \in C^{1,\alpha}(Q_1^-)$ and e any direction in space and time. Then u_e is Hölder continuous in $Q_{1-\delta}^-$ for any $0 < \delta < 1$.*

Proof. This proof follows the same ideas of Theorem 2.1.8 but we use Lemma 2.3.5 to control the growth of u near the free boundary, the parabolic maximum principle and the parabolic interior estimates (see [27]) instead of its elliptic counterparts. □

Chapter 3

The two membranes problem

The two membranes problem was first studied by Vergara-Caffarelli [26] in the context of variational inequalities to describe the equilibrium position of two elastic membranes in contact with each other that are not allowed to cross. He considered the linear elliptic case, in which the problem can be reduced to the classical obstacle problem by looking at the difference between the two functions representing the position of each membrane.

Nearly 35 years later, Silvestre [25] studied the problem for a nonlinear operator in divergence form. He obtained the optimal $C^{1,1}$ regularity of solutions together with a characterization of the regularity of the free boundary, that is the boundary of the set where the two functions coincide. The strategy in his proof was to show that the difference of the two functions satisfies an obstacle problem for the linearized operator, for which the regularity theory of the solutions and the free boundary are well known. An important remark is that in both of these cases the operator governing the behavior of each function is the same.

In a recent paper, Caffarelli, De Silva and Savin [19] considered the two membranes problem for (possibly nonlocal) different operators, i.e. they consider the case in which one of the membranes (say the lower one) satisfies an equation that has higher order with respect to the other one. Here, heuristically, the lower order operator can be treated as a perturbation and some regularity for the lower membrane is obtained. Regularity from the upper membrane can then be deduced by solving an obstacle problem (with the lower membrane as the obstacle) and obtaining estimates for solutions of nonlocal obstacle type problems in which the obstacle is not smooth. Repeating these arguments, the optimal regularity is achieved.

We also point out that the problem has been studied by several authors in the general case of N membranes, see [8], [23], [1].

Here, motivated by a model from mathematical finance, we consider a version of the two membrane problem for two different fully nonlinear operators. It is worth pointing out that, for the case of two different operators of the same order, the only result available is the Hölder regularity obtained in [19]. In this chapter, we prove $C^{1,\alpha}$ regularity of the solution pair for (concave or convex) operators satisfying a compatibility condition (see Equation (3.4)) and $C^{1,1}$ regularity for the case of the Pucci extremal operators, which is optimal. Moreover, we give an explicit example that shows that no regularity can be expected to hold for the free boundary in general.

3.1 Notation and preliminaries

Throughout this chapter the ellipticity constants $\lambda, \Lambda \in \mathbb{R}$ will be fixed and will satisfy $0 < \lambda < \Lambda$. Given these, we denote by $\mathcal{M}^+$ and $\mathcal{M}^-$ the Pucci extremal operators with respect to the class of symmetric matrices whose eigenvalues lie between λ and Λ, that is for any symmetric matrix X

$$\mathcal{M}^+(X) = \sup_{A \in \mathcal{L}_{\lambda,\Lambda}} \mathrm{tr}(AX) \qquad \text{and} \qquad \mathcal{M}^-(X) = \inf_{A \in \mathcal{L}_{\lambda,\Lambda}} \mathrm{tr}(AX)$$

where

$$\mathcal{L}_{\lambda,\Lambda} = \{A \in \mathbb{R}^{n \times n} : A \text{ is symmetric and } \lambda Id \leq A \leq \Lambda Id\}$$

$X \geq Y$ meaning as usual that $X - Y$ is a positive semidefinite matrix.

Recall that an operator $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ is said to be uniformly elliptic with repect to the class $\mathcal{L}_{\lambda,\Lambda}$ if it satisfies

$$\mathcal{M}^-(X - Y) \leq F(X) - F(Y) \leq \mathcal{M}^+(X - Y) \tag{3.1}$$

for any pair of symmetric matrices X and Y.

We will assume without loss of generality that $F(0) = 0$. A useful remark that follows from (3.1) is that if u is a function satisfying $F(D^2u) = f$ then

$$\begin{cases} \mathcal{M}^+(D^2u) & \geq & f \\ \mathcal{M}^-(D^2u) & \leq & f. \end{cases} \tag{3.2}$$

In particular, u is a subsolution and a supersolution of two (possibly different) elliptic equations with bounded measurable coefficients.

3.2 Statement of the problem

The problem we will consider is the following: given two functions $u_0, v_0 \in C^\gamma(\partial B_1)$ and $f, g \in C^\gamma(B_1)$ for some $\gamma \in (0, 1)$, we want to study the solutions u and v of

$$
\begin{cases}
u & \geq & v & \text{in } B_1 \\
F(D^2u) & \leq & f & \text{in } B_1 \\
G(D^2v) & \geq & g & \text{in } B_1 \\
F(D^2u) & = & f & \text{in } B_1 \cap \Omega \\
G(D^2v) & = & g & \text{in } B_1 \cap \Omega \\
u & = & u_0 & \text{on } \partial B_1 \\
v & = & v_0 & \text{on } \partial B_1
\end{cases}
\tag{3.3}
$$

where

$$\Omega := \{u > v\},$$

F is convex and

$$G(X) = -F(-X). \tag{3.4}$$

Note that G thus defined will be concave and that particular examples are

$$F(D^2w) = \sup_{\alpha\in\Sigma} \operatorname{tr}(A_\alpha D^2w) \qquad \text{and} \qquad G(D^2w) = \inf_{\alpha\in\Sigma} \operatorname{tr}(A_\alpha D^2w)$$

with Σ some set of indexes and $A_\alpha \in \mathcal{L}_{\lambda,\Lambda}$ for every α. If A_α can be any matrix in $\mathcal{L}_{\lambda,\Lambda}$ then $F = \mathcal{M}^+$ and $G = \mathcal{M}^-$. It is in this latter case that we prove the optimal regularity. Note that the strict inequality assumed for the ellipticity constants avoids these operators to become just a multiple of the Laplacian.

Equation (3.3) is to be understood in the viscosity sense. More precisely: if φ is a C^2 function in B_1 satisfying for some $x_0 \in B_1$

$$\varphi(x) \leq u(x) \quad \text{in} \quad B_1, \qquad \varphi(x_0) = u(x_0)$$

(i.e. φ *touches* u *by below* at x_0) then

$$F(D^2\varphi(x_0)) \leq f(x_0).$$

Similarly, if φ touches v by above and of course the opposite inequalities (last two equations in (3.3)) hold if φ touches u by above (or v by below) in Ω. A simple remark that will be useful is that it is equivalent to use paraboloids instead of general C^2 functions.

Note that the convexity of F as well as the Hölder regularity for f and g are natural assumptions if we want to get optimal regularity. In fact, one expects the solutions to this problem to be $C^{1,1}$ as long as the equation they solve on the non-contact set is "good enough", meaning that we have at least $C^{1,1}$ regularity for it. This, in principle, is not true in general if F is not convex or f and g are merely bounded. Also, for the problem to make sense, we will assume throughout this chapter that $u_0 > v_0$ on ∂B_1. Moreover, we will assume that $f - g \geq 0$. Notice that if this was not the case the problem could lose interest and degenerate into just two independent fully nonlinear equations. Indeed, if $f - g < 0$, we would have (see (3.2))

$$\begin{cases} \mathcal{M}^+(D^2(v-u)) & > & 0 & \text{in } B_1 \\ v - u & < & 0 & \text{in } \partial B_1 \end{cases}$$

and due to the maximum principle $u > v$ in B_1. Then there is no contact set and we just have the respective equations for u and v.

Equation (3.3) models a so-called "bid and ask" situation in which we have an asset, a seller (represented by u) and a buyer (represented by v). The price of the asset is random and the transaction will only take place when u and v "agree on a price", i.e. when $u = v$. Moreover, we want to model the expected earnings of u and v, assuming that their strategy is optimal.

One can think of this problem as having two different (although related) features: on one hand, we have an "obstacle type" situation, in which u tries to maximize gain with v being an obstacle and vice versa (v minimizing cost and u being an obstacle), hence the constraint $u \geq v$, but perhaps it is more interesting the relationship between u and v itself. Because of the "bid and ask" nature of the model, the Bellman type equations that govern the behavior of our solutions are closely related (recall $F(X) = -G(-X)$) and it is precisely this feature which opens a way to get regularity even though the operators are different.

The main result of this chapter is the following:

Theorem 3.2.1. *Let u and v solve* (3.3) *in the viscosity sense with $F = \mathcal{M}^+$ and $G = \mathcal{M}^-$. Then u and v are $C^{1,1}$ in $B_{1/4}$ and*

$$\|D^2 u\|_{L^\infty(B_{1/4})}, \|D^2 v\|_{L^\infty(B_{1/4})} \leq C$$

where C depends only on $n, \lambda, \Lambda, \|f\|_{C^\gamma(B_1)}, \|g\|_{C^\gamma(B_1)}, \|v\|_{L^\infty(B_1)}$ and $\|u\|_{L^\infty(B_1)}$.

3.3 Existence

In this section, we prove the existence of solutions for our problem. We use the method of penalization, i.e. we are going to consider a family of unconstrained "penalized equations" whose solutions are uniformly bounded in some Hölder space and hence convergent up to a subsequence. Then we are going to show that the limit of that subsequence is actually a solution to (3.3) (see [18]).

The penalized problem we are going to consider is the following:

$$\begin{cases} F(D^2u_\varepsilon) &= f + \beta_\varepsilon(u_\varepsilon - v_\varepsilon) \quad \text{in } B_1 \\ G(D^2v_\varepsilon) &= g - \beta_\varepsilon(u_\varepsilon - v_\varepsilon) \quad \text{in } B_1 \\ u_\varepsilon &= u_0 \quad \text{in } \partial B_1 \\ v_\varepsilon &= v_0 \quad \text{in } \partial B_1 \end{cases} \tag{3.5}$$

where for each $\varepsilon > 0$ we define

$$\beta_\varepsilon(t) = \beta(t/\varepsilon) \tag{3.6}$$

with $\beta : \mathbb{R} \to \mathbb{R}$ a smooth function satisfying

$$-N \le \beta \le 0, \quad \beta' \ge 0, \quad \beta(t) = 0 \text{ when } t \ge 1, \quad \beta(t) = -N \text{ when } t \le 0, \tag{3.7}$$

and

$$N := \|f - g\|_{L^\infty(B_1)}. \tag{3.8}$$

To get the existence and a priori bounds for solutions of (3.5) we will use a fixed point argument. Hence, we will need global regularity results for equations of the form

$$\begin{cases} H(D^2u) &= h \quad \text{in } B_1 \\ u &= u_0 \quad \text{in } \partial B_1 \end{cases} \tag{3.9}$$

where H is a uniformly elliptic operator. Here we mostly follow Chapter 4 of [6] but since the proofs need to be modified slightly we sketch them below for completeness:

Proposition 3.3.1. *Let u be a viscosity solution of (3.9) with $h \in L^\infty(B_1)$ and $u_0 \in C^\gamma(\partial B_1)$. Then for any $x_0 \in \partial B_1$ we have*

$$\sup_{x \in B_1} \frac{|u(x) - u(x_0)|}{|x - x_0|^{\gamma/2}} \le C \tag{3.10}$$

where C is a constant depending only on $n, \lambda, \Lambda, \gamma, \|u_0\|_{C^\gamma(\partial B_1)}$ and $\|h\|_{L^\infty(B_1)}$.

Proof. We translate, rotate and add a constant to u so that it is defined on $B := B_1(1, 0, .., 0)$, $x_0 = (0, 0, ..., 0)$ and $u(0) = 0$. We want to show

$$\sup_{x \in B_1} \frac{|u(x)|}{|x|^{\gamma/2}} \leq C \tag{3.11}$$

For this let us define the barrier $\psi(x) = Cx_1^{\gamma/2}$ with C a constant to be determined. Notice that

$$|x|^\gamma = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{\gamma}{2}} = (2x_1)^{\frac{\gamma}{2}}$$

on ∂B and hence

$$u(x) = u_0(x) \leq [u_0]_{C^\gamma(\partial B_1)} |x|^\gamma \leq Cx_1^{\gamma/2} = \psi(x)$$

there. On the other hand, ψ satisfies

$$H(D^2\psi) \leq \mathcal{M}^+(D^2\psi) = C\lambda \frac{\gamma}{2}(\frac{\gamma}{2} - 1)x_1^{\frac{\gamma}{2}-2} \leq -\|h\|_{L^\infty(B_1)}$$

in B in the viscosity sense if we take C large enough. From the maximum principle it follows that $u \leq \psi$ in B.

Symmetrically, we see that $u \geq -\psi$ on ∂B and $H(D^2(-\psi)) \geq \|h\|_{L^\infty(B_1)}$ in B, so using the maximum principle again we get $u \geq -\psi$ and hence (3.11). $\square$

Now we prove the global Hölder estimates.

Proposition 3.3.2. *Let u be a viscosity solution of* (3.9) *with $h \in L^\infty(B_1)$ and $u_0 \in C^\gamma(\partial B_1)$. Then*

$$\|u\|_{C^\eta(B_1)} \leq C \tag{3.12}$$

where C is a constant depending only on $n, \lambda, \Lambda, \|u_0\|_{C^\gamma(\partial B_1)}$ and $\|h\|_{L^\infty(B_1)}$ and $\eta \leq \gamma/2$.

Proof. We start by recalling that by interior estimates (Proposition 4.10 in [6]) solutions of (3.9) are in $C^\alpha_{loc}(B_1)$ for some $\alpha > 0$. Let $\eta = \min\{\gamma/2, \alpha\}$, $x_1, x_2 \in B_1$ $r = |x_1 - x_2|$, and take $x_1', x_2' \in \partial B_1$ such that

$$d_1 := d(x_1, \partial B_1) = |x_1 - x_1'| \quad \text{and} \quad d_2 := d(x_2, \partial B_1) = |x_2 - x_2'|.$$

We assume without loss of generality that $d_2 \leq d_1$ and we want to show that

$$|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^\eta. \tag{3.13}$$

We split the proof in two cases:

Case 1: When $r \leq \frac{d_1}{2}$, from the rescaled version of the interior estimates applied to $u - u(x_1')$ on $B_{d_1}(x_1)$ and our estimates at the boundary (3.10) we have

$$\begin{aligned} d_1^\alpha \|u - u(x_1')\|_{C^\alpha(B_{d_1/2}(x_1))} &\leq C(d_1^2 \|h\|_{L^\infty(B_{d_1}(x_1))} + \|u - u(x_1')\|_{L^\infty(B_{d_1}(x_1))}) \\ &\leq C d_1^2 \|h\|_{L^\infty(B_{d_1})} + C d_1^{\gamma/2} \leq C d_1^{\gamma/2}. \end{aligned}$$

On the other hand

$$d_1^\eta \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\eta} \leq d_1^\alpha \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} \leq d_1^\alpha \|u(\cdot) - u(x_1')\|_{C^\alpha(B_{d_1/2}(x_1))}$$

so

$$\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\eta} \leq C d_1^{\gamma/2-\eta} \leq C$$

as desired in this case.

Case 2: When $r > \frac{d_1}{2}$, again by the boundary estimates (3.10) and the triangle inequality

$$\begin{aligned}
|u(x_1) - u(x_2)| &\leq |u(x_1) - u(x_1')| + |u(x_1') - u(x_2')| + |u(x_2') - u(x_2)| \\
&\leq C d_1^{\gamma/2} + C|x_1' - x_2'|^\gamma + C d_2^{\gamma/2} \leq C(d_1^{\gamma/2} + r^\gamma + d_1^\gamma + d_2^\gamma + C d_2^{\gamma/2}) \\
&\leq C r^{\gamma/2} = C|x_1 - x_2|^{\gamma/2}.
\end{aligned}$$

□

In our next proof we are also going to use global Hölder estimates for "equations with bounded measurable coefficients":

Proposition 3.3.3. *Let u be a viscosity solution of*

$$\begin{cases} \mathcal{M}^+(D^2u) \geq -\alpha & \text{in } B_1 \\ \mathcal{M}^-(D^2u) \geq \alpha & \text{in } B_1 \\ u = u_0 & \text{in } \partial B_1 \end{cases} \tag{3.14}$$

for α a positive constant and $u_0 \in C^\gamma(\partial B_1)$. Then

$$\|u\|_{C^\eta(B_1)} \leq C \tag{3.15}$$

where C is a constant depending only on $n, \lambda, \Lambda, \|u_0\|_{C^\gamma(\partial B_1)}$ and α and $\eta \leq \gamma/2$.

Proof. The proof follows exactly as that of Proposition 3.3.2. We just point out that in order to prove the boundary estimates (3.10) a barrier argument for the Pucci extremal operators is used that is trivially adapted to a situation like (3.14). This is then combined with interior estimates that also hold for (3.14) (see [6]) to give (3.15) $\square$

Finally, the following observation is going to be useful: it follows from the proof of Proposition 3.3.1 that the dependence of the constant on $\|h\|_{L^\infty(B_1)}$ is continuous. The same is true for the interior estimates (again, see [6]) and hence for the constant in (3.15).

We can now prove existence of the penalized problem:

Proposition 3.3.4. *Let $\beta : \mathbb{R} \to \mathbb{R}$ be a smooth bounded function, $u_0, v_0 \in C^\gamma(\partial B_1)$ and $f, g \in L^\infty(B_1)$. There exist $u, v \in C(B_1)$ such that*

$$\begin{cases} F(D^2u) &= f(x) + \beta(u - v) & \text{in } B_1 \\ G(D^2v) &= g(x) - \beta(u - v) & \text{in } B_1 \\ u &= u_0 & \text{in } \partial B_1 \\ v &= v_0 & \text{in } \partial B_1 \end{cases} \tag{3.16}$$

in the viscosity sense.

Proof. We will use Schauder's Fixed Point Theorem (see [14]). Let $\tilde{\alpha} = \eta/2$ with η as in Proposition 3.3.2 and consider the map

$$T : C^\eta(B_1) \times C^\eta(B_1) \longrightarrow C^\eta(B_1) \times C^\eta(B_1)$$

defined as $T(\bar{u}, \bar{v}) := (u, v)$, u, v satisfying

$$\begin{cases} F(D^2u) &= f(x) + \beta(\bar{u} - \bar{v}) & \text{in } B_1 \\ G(D^2v) &= g(x) - \beta(\bar{u} - \bar{v}) & \text{in } B_1 \\ u &= u_0 & \text{on } \partial B_1 \\ v &= v_0 & \text{on } \partial B_1 \end{cases} \tag{3.17}$$

Such u and v exist by Perron's method. Also, let

$X :=$

$\{(\bar{u}, \bar{v}) \in C^{\tilde{\alpha}}(B_1) \times C^{\tilde{\alpha}}(B_1) : \bar{u} = u_0, \bar{v} = v_0 \text{ on } \partial B_1, \|\bar{u}\|_{C^\eta(B_1)}, \|\bar{v}\|_{C^\eta(B_1)} \leq C\}$

with C to be determined later.

If we can show that X is a compact convex set in $C^\eta(B_1) \times C^\eta(B_1)$, that T is continuous in X and $T(X) \subset X$ then by Schauder's Fixed Point Theorem there exists a solution to (3.16). We divide the proof in several steps:

Step 1: convexity and compactness of X.

The convexity is trivial. As for the compactness, it is a straightforward consequence of the Arsela-Ascoli theorem since $\tilde{\alpha} < \eta$.

Step 2: $T(X) \subset X$.

Notice that if $\bar{u}, \bar{v} \in C^{\tilde{\alpha}}(B_1)$ and we let

$$h := f + \beta(\bar{u} - \bar{v}) \qquad \text{and} \qquad k := g - \beta(\bar{u} - \bar{v})$$

we have, as f, g, β are bounded, that $h, k \in L^\infty(B_1)$. Hence, from Proposition 3.3.2 we know that

$$\begin{aligned} \|u\|_{C^\eta(B_1)} &\leq C \\ \|v\|_{C^\eta(B_1)} &\leq C \end{aligned} \tag{3.18}$$

for some constant $C > 0$ that depends only on n, λ, Λ, $\|f\|_{L^\infty(B_1)}$, $\|g\|_{L^\infty(B_1)}$, $\|u_0\|_{C^\gamma(B_1)}$ and $\|v_0\|_{C^\gamma(B_1)}$ (this is the constant C used to define X). In particular, this implies that $(u, v) \in X$.

Step 3: T is continuous.

Let $T(\bar{u}, \bar{v}) = (u', v')$ and $T(\bar{\bar{u}}, \bar{\bar{v}}) = (u'', v'')$. We want to show that given $\varepsilon > 0$ we can find $\delta > 0$ so that

$$\|\bar{u} - \bar{\bar{u}}\|_{C^{\tilde{\alpha}}(B_1)} < \delta \qquad \text{and} \qquad \|\bar{v} - \bar{\bar{v}}\|_{C^{\tilde{\alpha}}(B_1)} < \delta$$

imply

$$\|u' - u''\|_{C^{\tilde{\alpha}}(B_1)} < \varepsilon \qquad \text{and} \qquad \|v' - v''\|_{C^{\tilde{\alpha}}(B_1)} < \varepsilon.$$

Notice that

$$F(D^2u') - F(D^2u'') = \beta(\bar{u} - \bar{v}) - \beta(\bar{\bar{u}} - \bar{\bar{v}})$$

and from the definition of uniform ellipticity we have

$$\mathcal{M}^-(D^2(u' - u'')) \leq F(D^2u') - F(D^2u'') \leq \mathcal{M}^+(D^2(u' - u''))$$

in the viscosity sense. Now let $w := u' - u''$. From the previous two inequalities we have

$$\left\{ \begin{array}{rcll} \mathcal{M}^+(D^2w) & \geq & -\|\beta\|_{C^1(\mathbb{R})}(|\bar{u} - \bar{\bar{u}}| + |\bar{v} - \bar{\bar{v}}|) & \text{in } B_1 \\ \mathcal{M}^-(D^2w) & \leq & \|\beta\|_{C^1(\mathbb{R})}(|\bar{u} - \bar{\bar{u}}| + |\bar{v} - \bar{\bar{v}}|) & \text{in } B_1 \\ w & = & 0 & \text{on } \partial B_1, \end{array} \right. \tag{3.19}$$

so if we pick δ small enough so that $\|\beta\|_{C^1(B_1)}(|\bar{u} - \bar{\bar{u}}| + |\bar{v} - \bar{\bar{v}}|) \leq \delta_0$ for some $\delta_0 > 0$ to be chosen, we can rewrite (3.19) as

$$\left\{ \begin{array}{rcll} \mathcal{M}^+(D^2w) & \geq & -\delta_0 & \text{in } B_1 \\ \mathcal{M}^-(D^2w) & \leq & \delta_0 & \text{in } B_1 \\ w & = & 0 & \text{in } \partial B_1. \end{array} \right. \tag{3.20}$$

Then, by Proposition 3.3.3 and the observation following it (and choosing $\alpha = \delta_0$ in (3.14)) we can pick δ_0 small enough to get

$$\|u' - u''\|_{C^{\tilde{\alpha}}(B_1)} = \|w\|_{C^{\tilde{\alpha}}(B_1)} \leq \|w\|_{C^{\eta}(B_1)} \leq \varepsilon$$

as desired. The proof of $\|v' - v''\|_{C^{\tilde{\alpha}}(B_1)} \leq \varepsilon$ follows in an analog way.

$\square$

We now show the main result of this section, which is the existence of solutions of the two membranes problem (3.3).

Theorem 3.3.5. *There exist $u, v \in C(B_1)$ that solve (3.3) in the viscosity sense.*

Proof. Let N and $\beta : \mathbb{R} \to \mathbb{R}$ be as in (3.8) and (3.7) and define $\beta_\varepsilon(t)$ as in (3.6). Now, let $u_\varepsilon, v_\varepsilon$ solutions of Problem (3.5), i.e. $u_\varepsilon, v_\varepsilon$ satisfy

$$\begin{cases} F(D^2 u_\varepsilon) &= f + \beta_\varepsilon(u_\varepsilon - v_\varepsilon) \quad \text{in } B_1 \\ G(D^2 v_\varepsilon) &= g - \beta_\varepsilon(u_\varepsilon - v_\varepsilon) \quad \text{in } B_1 \\ u_\varepsilon &= u_0 \quad \text{in } \partial B_1 \\ v_\varepsilon &= v_0 \quad \text{in } \partial B_1 \end{cases}$$

By Proposition 3.3.4 such $u_\varepsilon, v_\varepsilon$ and exist. Moreover, notice that as

$$\|\beta_\varepsilon\|_{L^\infty(\mathbb{R})} = \|\beta\|_{L^\infty(\mathbb{R})} \leq N$$

and $f, g \in L^\infty(B_1)$, Proposition 3.3.2 gives us $\|u_\varepsilon\|_{C^\eta(B_1)}, \|v_\varepsilon\|_{C^\eta(B_1)} \leq C$ for some $C > 0$ that does not depend on ε. Hence, by Arsela-Ascoli (up to subsequences) $u_\varepsilon \to u$ and $v_\varepsilon \to v$ uniformly on B_1 for some $u, v \in C^{\tilde{\eta}}(B_1)$ where $\tilde{\eta} < \eta$. We claim that u and v solve (3.3).

We first want to see that $u \geq v$. Assume not, i.e. suppose the exists $x \in B_1$ such that

$$u(x) - v(x) = -\delta < 0.$$

From the uniform convergence we will have $u_\varepsilon(x) - v_\varepsilon(x) < -\delta/2$ for ε small enough. In particular, $u_\varepsilon - v_\varepsilon$ has to have a negative minimum at some point $y \in B_1$ (recall that on ∂B_1 we have $u_0 \geq v_0$). Moreover, $u_\varepsilon - v_\varepsilon$ satisfies, by the convexity of F and the fact that $F(X) = -G(-X)$,

$$F\Big(D^2\Big(\frac{u_\varepsilon - v_\varepsilon}{2}\Big)\Big) \leq \frac{1}{2}(f - g + 2\beta_\varepsilon(u_\varepsilon - v_\varepsilon))$$

in the viscosity sense. Let P be a plane touching $\frac{u_\varepsilon - v_\varepsilon}{2}$ by below at y. Then

$$F(D^2 P) \equiv 0$$

but since in particular P is a C^2 function we must have

$$F(D^2 P) \leq \frac{1}{2}(f(y) - g(y) + 2\beta_\varepsilon(P(y))) < 0$$

a contradiction.

Now we want to show that u and v satisfy the corresponding equations. Let us start by showing that $F(D^2 u) \leq f$ in B_1.

Let φ be a paraboloid touching u by below at $x_0 \in B_1$. Given $\xi > 0$ there exists $\delta > 0$ such that

$$f(x) \leq f(x_0) + \xi$$

for any $x \in B_\delta(x_0) \subset B_1$. Also, for any $\eta > 0$ we can choose ε small enough so that a translation of $\varphi(x) - \frac{\eta}{2}|x - x_0|^2$ (which we call $\tilde{\varphi}$) touches u_ε by below at $x_1 \in B_\delta(x_0)$. Hence

$$F(D^2\tilde{\varphi}(x_1)) \leq f(x_1) + \beta_\varepsilon(u_\varepsilon(x_1) - v_\varepsilon(x_1)) \leq f(x_1) \leq f(x_0) + \xi.$$

Since ξ was arbitrary we get

$$F(D^2\tilde{\varphi}(x_1)) \leq f(x_0)$$

but

$$F(D^2\tilde{\varphi}(x_1)) = F(D^2\varphi(x_1) - \eta \mathrm{Id}) = F(D^2\varphi(x_0) - \eta \mathrm{Id})$$

and letting $\eta \to 0$ we get

$$F(D^2\varphi(x_0)) \leq f(x_0)$$

as desired (recall that F is continuous in the space of matrices).

Using again the uniform convergence, the definition of viscosity solutions and considering a test function φ that touches u by above we can similarly show that $F(D^2u) \geq f$ in $B_1 \cap \Omega$ and conclude that $F(D^2u) = f$ in $B_1 \cap \Omega$. The proofs of $G(D^2v) \geq g$ in B_1 and $G(D^2v) = g$ in $B_1 \cap \Omega$ are analogous to the previous reasoning. It is immediate from uniform convergence that $u = u_0$ and $v = v_0$ on ∂B_1. $\square$

Remark 3.3.1. It is noting that the proof would still hold if we slightly relax the assumptions on the operators by asking just $F(X) \leq -G(-X)$.

Remark 3.3.2. We point out that there is no uniqueness in this problem. This comes from the fact that "there is no equation" on the contact set. In fact, let (for $n = 1$)

$$u(x) = \begin{cases} \frac{x_+^2}{2} & \text{for} \quad 0 < x \leq 1 \\ 0 & \text{for} \quad -1 \leq x \leq 0 \end{cases} \qquad v(x) = \begin{cases} -\frac{x_+^2}{2} & \text{for} \quad 0 < x \leq 1 \\ 0 & \text{for} \quad -1 \leq x \leq 0. \end{cases}$$

u and v are $C^{1,1}$ functions and they are strong solutions (and hence viscosity solutions) of (3.3) with $F = \mathcal{M}^+, G = \mathcal{M}^-, f = \Lambda$ and $g = -\Lambda$. However, we can make a perturbation $\psi \in C_c^\infty((-1,0))$ such that

$$-1 \leq \psi'' \leq 1$$

in $(-1, 0)$ and get another solution.

Of course this example can be easily generalized to $n \geq 2$ choosing

$$u(x) = \begin{cases} (|x| - 1/2)_+^2 & \text{in} \quad B_1 \setminus B_{1/2} \\ 0 & \text{in} \quad B_{1/2} \end{cases}$$

and

$$v(x) = \begin{cases} -(|x| - 1/2)_+^2 & \text{in} \quad B_1 \setminus B_{1/2} \\ 0 & \text{in} \quad B_{1/2} \end{cases}$$

and modifying the right hand sides accordingly.

However, uniqueness does hold in the "non-excercise region" Ω. In fact, if two pairs of solutions (u, v) and (u', v') satisfy

$$u \geq u' \quad \text{and} \quad v \geq v' \quad \text{on } \partial B_1 \cup \partial\Omega$$

then

$$u \geq u' \quad \text{and} \quad v \geq v' \quad \text{in } B_1 \cap \Omega$$

by the maximum principle for fully nonlinear elliptic equations (notice that in $B_1 \cap \Omega$ we have $F(D^2u) = f$ and $G(D^2v) = g$).

3.4 Regularity for the solution pair

In this section we prove our main regularity result, Theorem 3.2.1. The fact that this is the optimal regularity can be easily seen by considering $u(x) = \frac{x_+^2}{2}$ and $v(x) = -\frac{x_+^2}{2}$ (in one dimension) and noticing that they solve (3.3) with $f \equiv \Lambda$ and $g \equiv -\Lambda$ in $[-1, 1]$.

To prove Theorem 3.2.1 we show that solutions to (3.3) satisfy the hypothesis of the following Theorem (see Theorem 2.1 in [16]):

Theorem 3.4.1. *Let $f \in C^\gamma(B_1)$ and u a $W^{2,n}(B_1)$ solution of*

$$\begin{cases} F(D^2u) &= f(x) \quad \text{in } B_1 \cap \Omega \\ |D^2u| &\leq C \qquad \text{a.e. in } B_1 \cap \Omega^c \end{cases}$$

for some open set $\Omega \subset B_1$ and some elliptic operator F that is either concave or convex. Then, there exists a constant C depending only on $\|f\|_{C^\gamma(B_1)}$, $\|u\|_{W^{2,n}(B_1)}$, the dimension and the ellipticity constants such that

$$|D^2u| \leq C \quad \text{a.e. in } B_{1/2}.$$

The first step is to show the following Calderón-Zygmund type estimate:

Proposition 3.4.2. *Let u and v solve* (3.3). *Then u and v belong to $W^{2,p}(B_{1/2})$ for any $1 < p < \infty$ and*

$$\|u\|_{W^{2,p}(B_{1/2})}, \|v\|_{W^{2,p}(B_{1/2})} \leq C \tag{3.21}$$

for some constant C depending only on $n, \lambda, \Lambda, \|u\|_{L^\infty(B_1)}, \|v\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}$ and $\|g\|_{L^\infty(B_1)}$.

Proof. We prove the result for u, the proof for v is analogous. We will show that $|F(D^2u)| \leq C$ in the viscosity sense for some universal constant C. The result will then follow from Theorem 7.1 in in [6] (recall that $F(\cdot)$ is a convex operator).

Let φ be a C^2 function touching u by below at $x_0 \in B_1$. Recall that u is a viscosity supersolution across the whole ball (disregarding if x_0 is in the contact set or not), so we have

$$F(D^2\varphi(x_0)) \leq f(x_0) \leq \|f\|_{L^\infty(B_1)}.$$

If instead φ touches u by above, we separate two cases:

Case 1: if $x_0 \in \Omega$ then u is also a subsolution and we get

$$F(D^2\varphi(x_0)) \geq f(x_0) \geq -\|f\|_{L^\infty(B_1)}.$$

Case 2: if $x_0 \notin \Omega$, notice that φ also touches v by above, and v is a subsolution for G across the whole ball. Then

$$G(D^2\varphi(x_0)) \geq g(x_0) \geq -\|g\|_{L^\infty(B_1)}.$$

But for any symmetric matrix X we have $G(X) \leq F(X)$. Thus

$$F(D^2\varphi(x_0)) \geq g(x_0) \geq -\|g\|_{L^\infty(B_1)}$$

and we are done. $\square$

Remark 3.4.1. Notice that the proof is still valid if we just require $F(X) \geq G(X)$.

Now we show that when problem is given by the Pucci extremal operators solutions are $C^{1,1}$ on the contact set (i.e. they have bounded second derivatives). More precisely:

Proposition 3.4.3. *Let u and v solve* (3.3) *with $F = \mathcal{M}^+$ and $G = \mathcal{M}^-$. Then u and v are $C^{1,1}$ in $B_{1/2} \cap \Omega^c$ and*

$$\|D^2 u\|_{L^\infty(B_{1/2}\cap\Omega^c)}, \|D^2 v\|_{L^\infty(B_{1/2}\cap\Omega^c)} \leq C$$

for some universal constant C.

Proof. By Proposition 3.4.2 u and v are $W^{2,p}$ functions in, say, $B_{3/4}$ so we only need to show an almost everywhere bound on D^2u and D^2v in Ω^c. Also, since the notions of viscosity solution and strong solution coincide for $W^{2,p}$ with $p \geq n$ (see [20]) we have that (3.3) is satisfied a.e.

If x is a point in the interior of Ω^c for which (3.3) is satisfied, u and v coincide in a neighborhood of x and hence, letting e_u and e_v denote the eigenvalues of D^2u and D^2v respectively, we find

$$\begin{aligned} C \geq (f-g)(x) &\geq F(D^2u(x)) - G(D^2v(x)) \\ &= \Lambda \sum_{e_u>0} e_u + \lambda \sum_{e_u \leq 0} e_u - \lambda \sum_{e_v>0} e_v - \Lambda \sum_{e_v\leq 0} e_v \\ &= (\Lambda - \lambda) \sum_{e_v} |e_v| \end{aligned}$$

and the result follows in this case.

If $x \in \partial\Omega^c$ (again, a point at which (3.3) holds), $u-v$ has a minimum at x and hence $D^2(u-v)(x)$ is nonnegative definite, which in particular implies $\partial_{ee}u(x) \geq \partial_{ee}v(x)$ for any direction $e \in S^{n-1}$. Let us now pick a system of coordinates, say $\{e_1, \dots, e_n\}$, in which $D^2v(x)$ is diagonal. Moreover let us assume without loss of generality that the first m eigenvalues of D^2v are non-positive and the remaining $n-m$ positive. Let then A be a diagonal matrix with λ in the first m positions of its diagonal and Λ otherwise. Since A is a competitor in the sup and inf that define F and G respectively we have, using the equation,

$$\begin{aligned}
C \geq (f-g)(x) &\geq F(D^2u(x)) - G(D^2v(x)) \geq \mathrm{tr}(AD^2u(x)) - \mathrm{tr}(AD^2v(x)) \\
&= \lambda \sum_{i=1}^{m} u_{e_ie_i} + \Lambda \sum_{i=m+1}^{n} u_{e_ie_i} - \Lambda \sum_{i=1}^{m} v_{e_ie_i} - \lambda \sum_{i=m+1}^{n} v_{e_ie_i} \\
&\geq \lambda \sum_{i=1}^{m} v_{e_ie_i} + \Lambda \sum_{i=m+1}^{n} v_{e_ie_i} - \Lambda \sum_{i=1}^{m} v_{e_ie_i} - \lambda \sum_{i=m+1}^{n} v_{e_ie_i} \\
&= (\lambda - \Lambda) \sum_{e_v \leq 0} e_v + (\Lambda - \lambda) \sum_{e_v > 0} e_v \\
&= (\Lambda - \lambda) \sum_{e_v} |e_v|
\end{aligned}$$

so we get the bound for $D^2v(x)$. The proof of the bounds for $D^2u(x)$ is completely analogous. □

Finally, we can give the

Proof of Theorem 3.2.1. Again, we prove the result for u. Notice that, due to Proposition 3.4.2, u is $W^{2,n}(B_{3/4})$. Moreover, by Proposition 3.4.3 the Hessian of u is bounded a.e. inside the contact set in $B_{1/2}$, and hence we have

$$\begin{cases} F(D^2u) &= f(x) \quad \text{in } B_{1/2} \cap \Omega \\ |D^2u| &\leq C \quad\quad \text{in } B_{1/2} \cap \Omega^c \end{cases}$$

and we can apply Theorem 3.4.1 to get that $u \in C^{1,1}(B_{1/4})$ as desired. $\square$

3.5 The Free boundary

The classic approach to study the regularity of the free boundary of the double membrane problem consists on subtracting the two membranes (solutions), say $w := u - v$, and reduce the situation to an obstacle-type problem (note that w thus defined is nonnegative). One of the key steps of the analysis of the free boundary is to show that w satisfies a non-degeneracy property around free boundary points; that is, given $x_0 \in \partial\{w > 0\}$ one wants to show that

$$\sup_{\partial B_r(x_0)} w \geq Cr^2 \quad \text{for} \quad r > 0 \tag{3.22}$$

where $C > 0$ is a universal constant.

In the case of (3.3) this property is not satisfied. Indeed, let C be any positive constant and consider

$$u(x, y) := x^2 - y^2 + Cx_+^3 \quad \text{and} \quad v(x, y) := x^2 - y^2.$$

Here $x_+ = \max\{x, 0\}$. Notice that

$$\begin{cases} u &\geq v & \text{in } B_1 \\ \mathcal{M}^+(D^2u) &= 2(\Lambda - \lambda) + 6C\Lambda x_+ & \text{in } B_1 \\ \mathcal{M}^-(D^2v) &= -2(\Lambda - \lambda) & \text{in } B_1 \end{cases}$$

In particular u, v solve (3.3), $0 \in \partial\{w > 0\}$ and

$$\sup_{\partial B_r(0)} w = C \sup_{\partial B_r(0)} x_+^3 = Cr^3 < Cr^2 \tag{3.23}$$

for any $r < 1$.

In fact, by previous the following example, we can see that no free boundary regularity can hold in general. If we make

$$u(x, y) := x^2 - y^2 + \psi(x, y) \quad \text{and} \quad v(x, y) := x^2 - y^2$$

with ψ a nonnegative smooth function, we can make the contact set arbitrarily bad and still get solutions of (3.3).

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